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Stability of multiple-loop nonlinear time-varying systems

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CHAPTER 1: INTRODUCTION

Stability in Functional Analysis Setting

The central problem of stability theory is to ascertain qualitative features of system behavior in the absence of knowledge of specific system solutions. Typically an intuitive notion of the type of behavior desired is expressed in the form of a precise mathematical definition of stability. Then conditions on system parameters are sought which are sufficient to guarantee the system displays this type of stability. Here it is desired that a system be characterized by one of the following two types of behavior:

(1) The system is not explosive.

(2) The system is not critically sensitive to noise.

Concepts from functional analysis provide an appropriate vehicle for translating these notions of desired behavior into stability definitions.

First a suitable model of a system is required. Here a system is viewed in a "black box" sense as a mathematical relation which connects input functions from an input space with output functions from an output space. A relation is not explosive if inputs of finite "size" correspond only to outputs of finite "size". The notion of "size" is given a mathematical interpretation as a norm on a function space. Then a relation is not explosive if each set of bounded inputs corresponds to a set of bounded outputs. Such a relation is termed bounded. A problem occurs here due to the fact that the usual spaces from analysis contain only bounded functions. These spaces are unacceptable for use as output spaces since it would be required at the outset that bounded inputs lead to bounded outputs. This difficulty is obviated by employing an enlargement of the typical normed space, the extended space, as the space of input and output functions.

The other type of stable behavior considered here is lack of critical sensitivity to noise. Intuitively this type of behavior is displayed by a relation where inputs arbitrarily "close" to each other lead to outputs arbitrarily "close" to each other. This notion of stability is made precise by utilizing the norm of the difference of two functions in a function space as a measure of their "closeness". Then lack of critical sensitivity to noise is seen to be equivalent mathematically to continuity. Another useful physical interpretation of continuity is that it precludes the jump phenomenon.

Gain and Sector Conditions

Boundedness results are presented here in terms of gain or sector conditions on certain relations. The gain of a relation is roughly defined as the maximum ratio of the norm of the output to the norm of the input. This is an appealing definition in view of the notion of gain employed in the linear theory. The sector condition is a generalization of the gain condition which allows the boundedness results to find much wider application.

Incremental counterparts to gain and sector conditions are employed to arrive at continuity results. Loosely speaking, the incremental gain of a relation is defined as the maximum ratio of the norm of the deviation in the output to the norm of the deviation in the input. A generalization of the incremental gain condition leads to the incremental sector condition.

Interesting practical interpretations of sector conditions can be given for certain function spaces. For instance, consider relations having input and output spaces which are extensions of the space of square integrable functions. A memoryless nonlinearity satisfies a certain sector condition if its graph lies within a region of the plane enclosed by two lines passing through the origin. If the further restriction is made that the slope of the nonlinearity lies between two particular constants, then a certain incremental sector condition is satisfied. For a linear time-invariant system a sector condition and its incremental counterpart are equivalent. Further, it is found that if the Nyquist diagram is situated appropriately relative to a particular circle in the complex plane then a certain sector condition is satisfied.

Multiple-Loop System

The idea of a multiple-loop system is translated into precise mathematical terms as a set of simultaneous functional equations. A block diagram corresponding to these equations takes the form of an interconnection of a number of relations. The input supplied to each relation is composed of a general system input plus a weighted sum of outputs provided by other relations. The set of inputs and outputs of the relations whose interconnection produces the multiple-loop system is viewed as the set of general system outputs of the multiple-loop system. Stability of the multiple-loop system is interpreted in terms of stability of the collection of relations which connect the general system input with each of the general system outputs. If each of these relations is stable, then the multiple-loop system is referred to as stable.

Here stability results are presented in terms of the interconnection structure of a multiple-loop system and in terms of the gains of the relations which are interconnected. For a single-loop system these results lead to the intuitively appealing conclusion that an open-loop gain product less than unity implies closed-loop stability.

It is found that a certain transformation of a multiple-loop system allows gain stability conditions to be generalized to sector stability conditions. Due to this transformation, the theory finds much wider application than at first seems possible. One illustration of sector results is provided by considering a multiple-loop system which is an interconnection of an arbitrary number of memoryless nonlinearities with a number of linear time-invariant relations. For such a system, stability conditions can be found involving Nyquist diagrams of the linear parts and requiring the nonlinearities to satisfy certain sector conditions. In the case of a single loop these results reduce to previously obtained results [8], [16] reminiscent of the Nyquist criterion. Further manipulation leads to the familiar Popov conditions [16].

From the manner in which results are proven, it is clear that if the stability conditions are satisfied then bounds on system outputs or deviations in system outputs can actually be calculated. If tighter restrictions are placed on system parameters, then tighter bounds are obtained. In this sense the margin by which a system satisfies stability conditions is a measure of "how stable" that system is. Hence, some feeling can be obtained for the "degree of stability" of a multiple-loop system.

Single-Loop System

All stability results obtained prior to this investigation pertain only to a single-loop system. The results presented here are most closely related to results presented by Zames [15] and Sandberg [10]. In fact, it is found for the special case of a single-loop that the multiple-loop stability conditions specialize to conditions Zames [15] presents.

From a certain perspective the multiple-loop formulation is no more general than the single-loop. After all, any multiple-loop system can be represented as a single loop possessing open-loop elements which are multiple-input multiple-output. Then the single-loop theory applies. A disadvantage of this approach is that it tends to hide the influence the actual structure of the interconnection has on the problem. An advantage of the single-loop view is that fewer stability conditions are imposed on the system. However, these conditions are in general more difficult to verify than ones found from the multiple-loop view.

Outline

In Chapter 2 some necessary nomenclature is established. Prior results pertinent to this investigation are discussed in Chapter 3. In Chapter 4 new results are presented. A detailed description of what is meant here by a multiple-loop system is given followed by several stability theorems. Several applications of the theory are presented in Chapter 5. The conclusion is provided by Chapter 6.

CHAPTER 2: NOTATION

The symbol ε denotes set inclusion. Union is denoted by $\cup$ and intersection by $\cap$. The supremum and maximum are denoted, respectively, by sup and max. The symbol j is used for $\sqrt{-1}$ and s always denotes a complex number with real part σ and imaginary part ω. The conjugate of a complex number a is denoted by $\bar{a}$. R^n denotes an n-dimensional Euclidean space. For the vectors $x,y\varepsilon R^n$ the notation $x \leq y$ is used to indicate each component of x is less than or equal to the corresponding component of y.

The notation $f:X \to Y$ refers to the mapping f from the set X into the set Y. The notation $\{x:A\}$ is interpreted as the set of all x such that condition A is satisfied. The cartesian product of two spaces is defined by $X \times Y = \{(x,y):x\varepsilon X \text{ and } y\varepsilon Y\}$.

The identity matrix is denoted by I, a matrix with i,j^{th} element a_{ij} is denoted by $[a_{ij}]$, and a diagonal matrix with i^{th} diagonal element a_i is denoted by $[\text{diag } a_i]$. The transpose of the matrix A is denoted by A^T and the conjugate-transpose by A^*. The positive square root of the maximum eigenvalue of A^*A is denoted by $E\{A\}$. For a square matrix A, the determinant is denoted by $|A|$ and the inverse by A^{-1}.

Following the notation of [1], a minor of the matrix A is given by

$$A\begin{pmatrix} i_1, & i_2, & \dots, & i_p \\ k_1, & k_2, & \dots, & k_p \end{pmatrix} = \left| \begin{bmatrix} a_{i_1k_1} & a_{i_1k_2} & \cdots & a_{i_1k_p} \\ a_{i_2k_1} & a_{i_2k_2} & \cdots & a_{i_2k_p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_pk_1} & a_{i_pk_2} & \cdots & a_{i_pk_p} \end{bmatrix} \right|$$

where $i_1 < i_2 < \dots . < i_p$ and $k_1 < k_2 < \dots . < k_p$. The principal minors are those for which $i_j = k_j$ for each $j = 1,2,\dots,p$. For an $m \times m$ matrix A the successive principal minors are

$$A\begin{pmatrix}1\\1\end{pmatrix}, \; A\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \; \dots ., \; A\begin{pmatrix}1 & 2 & \dots . & m\\1 & 2 & \dots . & m\end{pmatrix}.$$

CHAPTER 3: PREVIOUS RESULTS

Relevant past investigations of stability in a functional analysis setting are discussed here. All prior efforts have been directed towards analysis of single-loop systems. The first result discussed deals with a system having an open loop composed of a multiple-input multiple-output linear time-invariant part in cascade with a bank of nonlinearities. For this system Sandberg [10] provides a stability result having a frequency-domain interpretation. Next some stability results concerned with boundedness and continuity of a certain pair of functional equations are discussed. These results which are due to Zames [15] are phrased in terms of gain and sector conditions. A discussion of several interpretations of sector conditions concludes this chapter.

There is one detail which should be made clear at the outset. All results discussed here are posed in such a manner as to separate questions of existence and uniqueness of solutions from questions of stability of solutions. This is certainly a logical separation and means for a stable system that whatever possibly nonunique solutions exist display the appropriate properties. Existence and uniqueness can often be established by use of the appropriate fixed point theorem [3], [4], [7].

A Frequency-Domain Result

Here a stability result is given for the system represented in Fig. 1. The block L represents a multiple-input multiple-output linear time-invariant system while the block N represents a bank of nonlinearities.

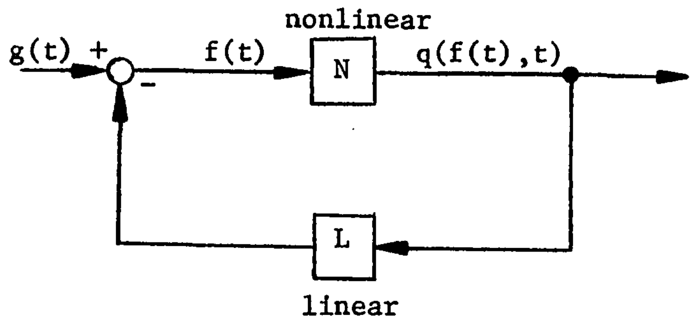


Fig. 1. Single-loop system having open loop composed of linear part in cascade with bank of nonlinearities.

First a suitable space for input and output functions is defined. Assume the functions to be dealt with are n-vector-valued functions of time. Consider the space

$$L_{2n}[0,\infty) = \{f:f \text{ is measurable and } \int_0^\infty f^T(t)f(t)dt<\infty\}.$$

The space desired is the extension of this space given by

$$E_n = \{f:f \text{ is measurable and } \int_0^t f^T(v)f(v)dv<\infty \text{ for all } t\varepsilon[0,\infty)\}.$$

Now N and L are described further and the system equations given. N is characterized by a function $q:R^n \times R \to R^n$ where

$$q(f(t),t) = [q_1(f_1(t),t),q_2(f_2(t),t)...,q_n(f_n(t),t)]^T$$

and the q_i are real-valued functions with the following properties for each i:

(1) $q_i(0,t) = 0$ for all $t\varepsilon[0,\infty)$.

(2) There exist real numbers a and b such that $a \leq \frac{q_i(w,t)}{w} \leq b$ for all $w \neq 0$ and all $t \varepsilon [0,\infty)$.

(3) $q_i(w(t),t)$ is a measurable function of t whenever w(t) is measurable.

L is characterized by an $n \times n$ matrix weighting function k(t). It is assumed each element of this matrix is in $L_1[0,\infty)$. Then L takes any input $u \varepsilon L_{2n}[0,\infty)$ into an output $h \varepsilon L_{2n}[0,\infty)$ by the integral equation

$$h(t) = \int_0^t k(t-v)u(v)dv.$$

It is not assumed L is described by an ordinary differential equation. However, if this is the case it appears at first that initial conditions can not be accounted for. This is not true because examination of the block diagram shows the negative of the initial condition response can be added to g and in this manner included in the analysis. Now the system equations represented by Fig. 1 are seen to be

$$g(t) = f(t) + \int_0^t k(t-v)q(f(v),v)dv.$$

The following theorem pertains to this system and is a special case of an abstract result presented by Sandberg [10].

Theorem 1: Let $g \varepsilon L_{2n}[0,\infty)$ and $f \varepsilon E_n$ satisfy the system equations. Define

$$K(s) = \int_0^\infty k(t)e^{-st}dt \text{ for } \sigma \geq 0.$$

Suppose that

(1) $|I + \frac{1}{2}(a + b)K(s)| \neq 0$ for $\sigma \geq 0$, and

(2) $\frac{1}{2}(b - a) \sup_{\omega} E\{[I + \frac{1}{2}(a + b)K(jw)]^{-1}K(jw)\} < 1.$

Then $f \varepsilon L_{2n}[0,\infty)$. |

It is interesting to observe that the <u>a priori</u> restriction of $f \varepsilon E_n$ is tantamount to assuming no finite escape time. The next theorem is also given in [10].

Theorem 2: Assume the hypotheses of Theorem 1 are satisfied, $g(t) \to 0$ as $t \to \infty$, and the elements of $k(t)$ are each in $L_2[0,\infty)$. Then $f(t) \to 0$ as $t \to \infty$. |

Sandberg shows that for n = 1 conditions (1) and (2) of Theorem 1 admit an interpretation in the complex plane. In [8] it is found that for $b > 0$ conditions (1) and (2) are satisfied if one of the following is true:

(1) For $a > 0$ the locus of $K(j\omega)$ for $\omega \varepsilon (-\infty,\infty)$ lies outside the circle with center $(-\frac{1}{2}(a^{-1} + b^{-1}),0)$ and radius $\frac{1}{2}(a^{-1} - b^{-1})$, and this locus does not encircle the point $(-\frac{1}{2}(a^{-1} + b^{-1}),0)$.

(2) For $a = 0$ the real part of $K(j\omega)$ is greater than $-b^{-1}$ for all ω.

(3) For $a < 0$ the locus of $K(j\omega)$ for $\omega \varepsilon (-\infty,\infty)$ is contained within the circle with center $(-\frac{1}{2}(a^{-1} + b^{-1}),0)$ and radius $\frac{1}{2}(b^{-1} - a^{-1})$.

The above are illustrated in Fig. 2 where the locus must lie in the shaded region for the appropriate condition to be true.

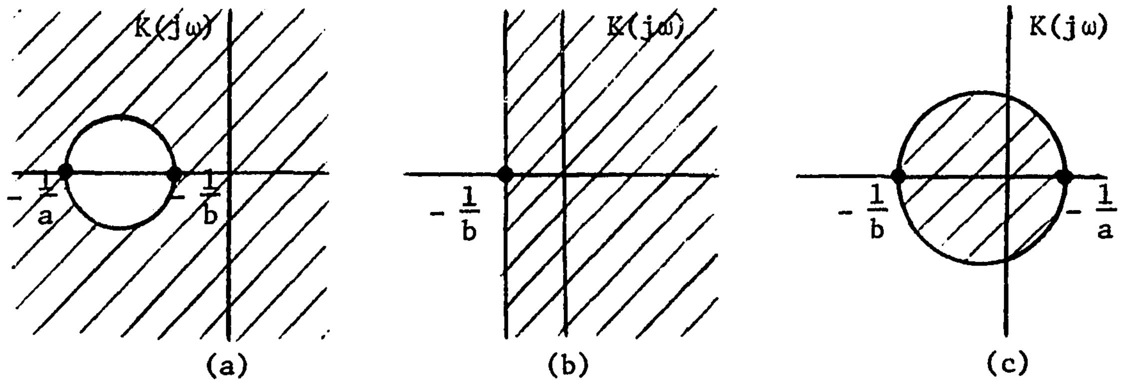


Fig. 2. Frequency-domain stability conditions.

(a) $a > 0$.

(b) $a = 0$.

(c) $a < 0$.

Now assume $n = 1$ and the block L is described by the ordinary differential equation

$$\sum_{j=0}^{m} a_j h^{(j)} = \sum_{j=0}^{m-1} b_j u^{(j)}$$

where the a_j and b_j are real constants, the superscript denotes the order of the differentiation, and $a_m \neq 0$. Assume there is no general input to the system so that g is the negative of the initial condition response of L. Further, assume the zeros of the polynomial $\sum_{j=0}^{m} a_j s^j$ are strictly in the left half plane. Then by Theorem 2, if a solution exists, all that is needed to infer $f(t) \to 0$ as $t \to \infty$ is that $K(jw)$ satisfy one of the conditions illustrated in Fig. 2 and the nonlinearity q satisfy the earlier listed conditions. Here then is an answer to the often asked question: Under what conditions will the system response go to zero from arbitrary initial conditions?

There is clearly much similarity between the above and the familiar Nyquist criterion from the linear theory. In fact, if a = b the above conditions are identical with the Nyquist criterion. This connection with the linear theory gives reason to believe that these results are at least in the right "ball park".

Further stability results involving frequency-domain conditions are obtained in [9] for an L_∞ type of stability. Also systems modeled by difference equations are considered. In [11] results are given which provide continuity, exponential bounds, and ultimate periodicity of system responses.

Stability Results Involving Gain and Sector Conditions

Boundedness and continuity results obtained for a certain pair of functional equations are discussed here. These results presented by Zames [15] are phrased in terms of gain and sector conditions. First a suitable space for input and output functions of time is defined. Then the precise mathematical model used for a system is discussed, and definitions are given for boundedness and continuity. Finally, definitions of gain and sector conditions are given and stability theorems presented.

All input and output functions are real-valued and defined on the time interval T which is of the form $[t_0,\infty)$ or $(-\infty,\infty)$. The notion of truncation of such functions is employed to define extensions of the usual spaces of analysis.

Definition: For a real-valued function x defined on T the truncation at time $t\varepsilon T$ is given by

$$x_t(\tau) = \begin{cases} x(\tau) & \text{for } \tau < t \\ 0 & \text{for } \tau \geq t \end{cases} .$$

Appendix A contains the definition of a normed linear space. The following definition deals with a special kind of normed linear space.

Definition: X is a space of real-valued functions on T possessing the following properties:

(1) X is a normed linear space where if $x\varepsilon X$ the norm of x is denoted by $||x||$.

(2) If $x\varepsilon X$ then $x_t\varepsilon X$ for all $t\varepsilon T$.

(3) If x is such that $x_t\varepsilon X$ for all $t\varepsilon T$, then

(a) $||x_t||$ is a nondecreasing function of $t\varepsilon T$, and

(b) $\lim_{t\to\infty} ||x_t||$ is finite if and only if $x\varepsilon X$ where

$$\lim_{t\to\infty} ||x_t|| = ||x|| \text{ if x does belong to X.}$$

Many of the common function spaces satisfy the conditions placed on X. For instance, these conditions are satisfied by the L_p spaces for $p = 1,2,\ldots,\infty$. Appendix A contains a discussion of L_p spaces. From the viewpoint of applications, the L_2 space of square integrable functions and the L_∞ space of bounded functions are of particular interest.

Now an extension of the space X is defined which serves as a suitable space for input and output functions in the stability problem formulation.

Definition: The extended space X_e is the linear space of real-valued functions of time each having all finite truncations in X. Thus,

$$X_e = \{x : x \text{ is a real-valued function on } T \text{ and } x_t \in X \text{ for all } t \in T\}.$$

An extended norm is defined for $x \in X_e$ by $||x||_e = ||x||$ if $x \in X$ and $||x||_e = \infty$ if $x \notin X$.|

It should be noted that despite the definition of the extended norm the linear space X_e is not a normed linear space.

Since the extended space contains "explosive" functions, it becomes a suitable space for inputs and outputs where X is not. Use of X for inputs and outputs would require knowing a priori that the system is not explosive. This would result in assuming stability to prove stability.

The definition of X_e makes the significance of assumptions (2) and (3) in the definition of X clear. Assumption (2) guarantees X_e is an enlargement of X by implying $X \subset X_e$. If $x \in X_e$ then assumption (3) allows determination of whether or not x has finite norm by examining $\lim_{t \to \infty} ||x_t||$. This fact is crucial to the proofs of stability theorems presented later.

The precise mathematical model of a system employed here is that of a relation defined below.

Definition: A relation H on X_e is a subset of the product space $X_e \times X_e$. If $(x,y) \varepsilon H$ then y is said to be an image of x under H and is often denoted by Hx. The notation $Hx(t)$ refers to the value of an image of x under H at time t. The domain of H is defined by

$$Do(H) = \{x: \text{ there exists a y so that } (x,y) \varepsilon H\},$$

and the range of H is defined by

$$Ra(H) = \{y: \text{ there exists an x so that } (x,y) \varepsilon H\}.$$

If A is a subset of X_e, the image of A under H is defined by

$$HA = \{y: (x,y) \varepsilon H \text{ and } x \varepsilon A \cap Do(H)\}.|$$

Appendix B contains further discussion of relations. If H and K are relations and c is a real constant, then the sum H + K, the product cH, and the composition product KH are defined in the usual way. Further, the inverse relation H^{-1} always exists, and the identity relation is denoted by I. A relation which is single-valued and has the entire X_e space as domain is termed an operator.

It is interesting to observe that use of X_e in defining a relation essentially requires the relation to have no finite "escape time". This is due to the fact that an output truncated at a finite time must have a finite norm.

Systems for which it makes sense to speak of initial conditions can be modeled as a relation in basically two ways. A single relation can be used which is multiple-valued having each output correspond to a different initial condition. An alternative is to use a different relation corresponding to each initial condition. If the system is linear and described by a set of ordinary differential equations, then a relation can be utilized to account only for the forced response and the initial condition response can be simply added to the output.

Now the stability properties of boundedness and continuity are defined.

Definition: A subset S of X_e is said to be bounded if there exists a real number D such that if $x \in S$ then $||x||_e < D$. A relation H on X_e is bounded if the image of every bounded subset of X_e is itself a bounded subset of X_e.|

Note this definition is stronger than simply saying the image of each input of finite norm is itself of finite norm. In the latter case it might be possible to have a sequence $x_n, n = 1,2,\ldots,$ with $||x_n|| = 1$ and $||Hx_n|| = n$ for each n.

Definition: A relation H on X_e is continuous if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in Do(H)$, $y \in Do(H)$, and $||x-y||_e < \delta$ then $||Hx-Hy||_e < \varepsilon$.|

It is interesting to observe that x-y can have a finite norm even if x and y both have infinite norms. This leads to the fact that for a continuous relation explosive inputs which are arbitrarily "close" to each other lead to outputs which are also arbitrarily "close" to each other. It is also interesting to observe that continuity is incompatible with the jump phenomenon. Further it should be noted that a continuous relation must be single-valued.

In [15] Zames investigates boundedness and continuity of the single-loop system illustrated in Fig. 3. The functional equations describing this system are

$$
\begin{aligned}
e_1 &= a_1x + w_1 + y_2 \\
e_2 &= a_2x + w_2 + y_1 \\
y_1 &= H_1e_1 \\
y_2 &= H_2e_2
\end{aligned}
\tag{1}
$$

where it is assumed that:

H_1 and H_2 are relations on X_e.

x in X_e is an input.

a_1 and a_2 are real constants.

w_1 and w_2 are fixed biases in X.

e_1, e_2, y_1, and y_2 all in X_e are outputs.

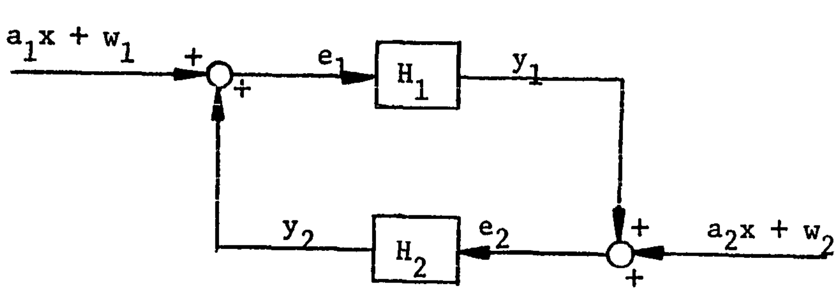


Fig. 3. Block diagram of single-loop system.

For stability purposes interest is focused on the relations which connect the input x to each of the outputs e_1, e_2, y_1 and y_2. These relations are designated by E_1, E_2, F_1 and F_2, respectively. E_1 is defined by

$$E_1 = \{(x,e_1):(x,e_1)\varepsilon X_e \times X_e \text{ and there exists } e_2,\ y_1,\ y_2,$$

$$H_1 e_1, \text{ and } H_2 e_2 \text{ such that (1) is true}\}.$$

The relations E_2, F_1, and F_2 are similarly defined.

From earlier discussion it is seen that the biases w_1 and w_2 can be used to account for initial condition responses.

At this point it is interesting to observe how the use of a relation as the basic system model makes it possible to avoid questions of existence and uniqueness of solutions. Examining E_1, for instance, it is clear the domain of E_1 is not required to be the entire X_e space. Hence, it is not required that there exist a solution corresponding to each input. Further, for an input x which is in the domain of E_1, it is not required the corresponding e_1 be unique.

Certainly in most problems it is desired that there exist unique solutions. However, it may be extremely difficult to mathematically determine this. By formulating the problem in terms of relations this poses no difficulty for the stability analysis. This sort of situation occurs, for instance, if the relation H_2 is a hysterises nonlinearity and the realtion H_1 is linear, time-invariant, and modeled by a set of ordinary differential equations.

Another reasonable approach to the stability problem is to use an operator as the basic system model. Since an operator is single-valued and has the entire X_e space as domain, this approach requires the a priori assumption of existence of unique solutions.

Now the notion of gain is made precise for a relation.

Definition: For a relation H with all $(Hx)_t = 0$ whenever $x_t = 0$, $x \varepsilon Do(H)$, and $t \varepsilon T$, the gain is defined by

$$g(H) = \sup \frac{||(Hx)_t||}{||x_t||}$$

where the supremum is taken over all $x \varepsilon Do(H)$, all $Hx \varepsilon Ra(H)$, and all $t \varepsilon T$ for which $x_t \neq 0$.|

The following inequalities are obtained directly from the definition of gain and are crucial to proofs of stability theorems:

$$||(Hx)_t|| \leq g(H)\ ||x_t|| \quad \text{for } x \varepsilon Do(H) \text{ and } t \varepsilon T,$$

$$||Hx||_e \leq g(H)\ ||x||_e \quad \text{for } x \varepsilon Do(H).$$

The second follows from the first on letting $t \rightarrow \infty$.

Now a theorem proven by Zames [15] which provides boundedness conditions for the single loop system is presented.

Theorem 3: The relations E_1, E_2, F_1, and F_2 associated with the single-loop system are bounded if $g(H_1)g(H_2) < 1$.|

The incremental counterpart of the definition of gain is supplied by the following definition.

Definition: For a relation H with all $(Hx-Hy)_t = 0$ whenever $(x-y)_t = 0$, x and $y \varepsilon Do(H)$, and $t \varepsilon T$, the incremental gain is defined by

$$\hat{g}(H) = \sup \frac{||(Hx-Hy)_t||}{||(x-y)_t||}$$

where the supremum is taken over all $x, y \varepsilon Do(H)$, all $Hx, Hy \varepsilon Ra(H)$, and all $t \varepsilon T$ for which $(x-y)_t \neq 0$.|

The following inequalities similar to those for the nonincremental case are satisfied:

$$||(Hx-Hy)_t|| \leq \hat{g}(H)||(x-y)_t|| \text{ for x and } y \varepsilon Do(H) \text{ and } t \varepsilon T,$$

$$||Hx-Hy||_e \leq \hat{g}(H) \; ||x-y||_e \text{ for x and } y \varepsilon Do(H).$$

Now a continuity theorem given in [15] is presented.

Theorem 4: The relations E_1, E_2, F_1, and F_2 associated with the single-loop system are continuous if $\hat{g}(H_1)\hat{g}(H_2) < 1$.|

It is often true that the problem can be presented in such a manner that zero is in the domains of both H_1 and H_2 and has a unique image of zero under both relations. In this situation the condition of Theorem 4 is also sufficient for boundedness. This is seen by setting y = 0 in the definition of incremental gain. Then it is clear that $\hat{g}(H_1)\hat{g}(H_2) < 1$ implies $g(H_1)g(H_2) < 1$.

A certain transformation of the single-loop system results in a significant generalization of Theorems 3 and 4. It is found many more systems can be examined than at first appears possible. The effect of the transformation on stability conditions is to change them from gain restrictions to conicity restrictions. Following is the definition of a conic relation.

Definition: A relation H on X_e is interior conic with center parameter c and radius parameter $r \geq 0$ if

$$||(Hx)_t - cx_t|| \leq r||x_t||$$

for all $x \varepsilon Do(H)$, all $Hx \varepsilon Ra(H)$, and all $t \varepsilon T$. H is exterior conic with center parameter c and radius parameter $r \geq 0$ if the above inequality is reversed.|

If X is an inner product space another notation defined below can be employed to specify the nature of a conic relation. The definition of an inner product space is given in Appendix A.

Definition: Assume H is a relation on the extension of an inner product space. H is inside the sector {a,b} if $a \leq b$ and

$$<(Hx)_t - ax_t, (Hx)_t - bx_t> \leq 0$$

for all $x \varepsilon Do(H)$, $Hx \varepsilon Ra(H)$, and $t \varepsilon T$. H is outside the sector {a,b} if the inequality is reversed.|

For the special case of an inner product space the specific correspondence between conicity conditions and sector conditions is indicated by the following two statements. A relation H is interior (exterior) conic with center parameter c and radius parameter r if H is inside (outside) the sector {c-r, c+r}. Conversely, a relation H is inside (outside) the sector {a,b} if H is interior (exterior) conic with center parameter $\frac{1}{2}(a+b)$ and radius parameter $\frac{1}{2}(b-a)$.

It is of interest to consider the situation where b goes to infinity in the definition of a sector. For a = 0 this limiting case is covered by the following definition of positivity.

Definition: A relation H on the extension of an inner product space is positive if

$$\langle x_t, (Hx)_t \rangle \geq 0$$

for all $x \varepsilon Do(H)$, all $Hx \varepsilon Ra(H)$, and all $t \varepsilon T$.|

An incremental counterpart for each of the three preceding definitions is provided by the following definitions.

Definition: A relation H on X_e is incrementally interior conic with center parameter c and radius parameter $r \geq 0$ if

$$||(Hx-Hy)_t - c(x-y)_t|| \leq r||(x-y)_t||$$

for all $x,y \varepsilon Do(H)$, all $Hx,Hy \varepsilon Ra(H)$, and all $t \varepsilon T$. H is incrementally exterior conic with center parameter c and radius parameter $r \geq 0$ if the above inequality is reversed.|

Definition: Assume H is a relation on the extension of an inner product space. H is incrementally inside the sector {a,b} if $a \leq b$ and

$$<(Hx-Hy)_t - a(x-y)_t, (Hx-Hy)_t - b(x-y)_t> \leq 0$$

for all $x,y \varepsilon Do(H)$, all $Hx,Hy \varepsilon Ra(H)$, and all $t \varepsilon T$. H is incrementally outside the sector $\{a,b\}$ if the above inequality is reversed.|

Definition: Assume H is a relation on the extension of an inner product space. H is incrementally positive if

$$<(x-y)_t, (Hx-Hy)_t> \geq 0$$

for all $x,y \varepsilon Do(H)$, all $Hx,Hy \varepsilon Ra(H)$, and all $t \varepsilon T$.|

It is easily found that for the special case of an inner product space the same type of correspondence exists between the incremental versions of conicity and sector conditions as for the nonincremental versions.

Now two theorems are presented which provide sufficient boundedness and continuity conditions phrased in terms of sector conditions.

Theorem 5: Let the open-loop relations H_1 and H_2 of the single-loop system be conic. Suppose for constants γ and ε where one is positive and one is zero that

(1) $-H_2$ is inside the sector $\{a+\gamma, b-\gamma\}$ where $b > 0$, and

(2) H_1 satisfies one of the following conditions:

Case 1a: If $a > 0$ then H_1 is outside the sector

$$\{-\frac{1}{a} - \varepsilon, -\frac{1}{b} + \varepsilon\}.$$

Case 1b: If $a < 0$ then H_1 is inside the sector

$$\{-\frac{1}{b} + \varepsilon, -\frac{1}{a} - \varepsilon\}.$$

Case 2: If $a = 0$ then $H_1 + (\frac{1}{b} - \varepsilon)I$ is positive

and if $\gamma = 0$ then $g(H_1) < \infty$.

Then the relations E_1, E_2, F_1, and F_2 associated with the single-loop system are each bounded.|

Theorem 6: Suppose all hypotheses of Theorem 5 are replaced by their incremental counterparts. Then the relations E_1, E_2, F_1, and F_2 associated with the single-loop system are each continuous.|

For the special case of an inner product space it is easily found that the gain theorems can be obtained from the sector theorems. To show this assume $g(H_1)g(H_2) < 1$. Theorem 5 can be utilized to find boundedness is implied. In this manner the results of Theorem 3 are obtained. First note that from

$$||(H_1x)_t|| \leq g(H_1)\ ||x_t|| \text{ for all } x\varepsilon Do(H_1), \text{ all } H_1x\varepsilon Ra(H_1), \text{ and all } t\varepsilon T$$

it is inferred that H_1 is interior conic with center parameter zero and radius parameter $g(H_1)$. Now this implies H_1 is inside $\{-g(H_1), g(H_1)\}$. Similarly it is found that $-H_2$ is inside $\{-g(H_2), g(H_2)\}$. Now define ε by $\varepsilon = \frac{1}{g(H_2)} - g(H_1)$. ε is positive since $g(H_1)g(H_2) < 1$.

This results in H_1 being inside $\{-\frac{1}{g(H_2)} + \varepsilon, \frac{1}{g(H_2)} - \varepsilon\}$. But setting $\gamma = 0$ in Theorem 5, it is seen from Case 1b that the relations E_1, E_2, F_1, and F_2 associated with the single-loop system are bounded. Using similar reasoning it is found that Theorem 4 can be obtained from Theorem 6.

Interpretations of Sector Conditions

Several interpretations of sector conditions for particular types of relations are available. Here some results presented by Zames [15], [16] are discussed. For a certain class of linear time-invariant operators on $L_{2e}[0,\infty)$ it is found sector conditions can be phrased in terms of conditions imposed on the Nyquist diagram. In general, for any relation on $L_{2e}[0,\infty)$, it is found certain conditions having an interpretation in the output versus input plane are sufficient for satisfaction of sector conditions. These conditions referred to as instantaneous sector conditions find particular application to memoryless nonlinearities, nonlinearities which are time varying, and hysteresis nonlinearities.

First consider the class of linear time-invariant operators defined below.

Definition: Q is the class of operators on $L_{2e}[0,\infty)$ satisfying an equation of the type

$$Hx(t) = h_\infty x(t) + \int_0^t h(t-v)x(v)dv$$

where h_∞ is a constant, the impulse response $h \in L_1[0,\infty)$, and for some $\sigma_o < 0$ the function $h(t)e^{-\sigma_o t}$ lies in $L_1[0,\infty)$.|

The Laplace transform of members of Q provides the means by which sector conditions can be interpreted in the complex plane.

Definition: The Laplace transform $\overline{H}(s)$ of $H \in Q$ is given by

$$\overline{H}(s) = h_\infty + \int_0^\infty h(t)e^{-st}dt \text{ for } \sigma \geq 0.|$$

Of course any linear time-invariant system modeled by a set of ordinary differential equations possesses a Laplace transform as defined above. Further, this is regardless of whether or not the impulse response lies in $L_1[0,\infty)$, but the transform may not be defined for all $\sigma \geq 0$. It is easily found that such a system has a corresponding integral equation in the class Q if and only if the poles of the Laplace transform lie strictly in the left half complex plane.

The following lemma proven in [16] phrases sector conditions in terms of the behavior of the Nyquist diagram in the complex plane.

Definition: The Nyquist diagram of $H \in Q$ is the locus of $\overline{H}(j\omega)$ for $\omega \in (-\infty,\infty)$.|

Lemma 1: Let H be an operator in Q and let c and $r \geq 0$ be constants.

(1) If $\overline{H}(s)$ satisfies the inequality

$$|\overline{H}(j\omega) - c| \leq r \text{ for all } \omega \in (-\infty,\infty),$$

then H is incrementally interior conic with center parameter c and radius parameter r.

(2) If $\bar{H}(s)$ satisfies the inequality

$$|\bar{H}(j\omega) - c| \geq r \quad \text{for all } \omega\varepsilon(-\infty,\infty)$$

and if the Nyquist diagram of H does not encircle the point (c,0), then H is incrementally exterior conic with center parameter c and radius parameter r.

(3) If $\text{Re}\{\bar{H}(j\omega)\} \geq 0$ for all $\omega\varepsilon(-\infty,\infty)$, then H is incrementally positive.|

Due to the linearity of relations in Q, the incremental and non-incremental sector conditions become equivalent. Hence, Lemma 1 is true in the nonincremental case also.

Now sector conditions are given an interpretation in the output versus input plane through the following definition of instantaneous sector conditions.

Definition: Assume H is a relation on $L_{2e}[0,\infty)$. Each of the following must be true for all $x\varepsilon Do(H)$, all $Hx\varepsilon Ra(H)$, and all $t\varepsilon T$.

(1) H is instantaneously inside the sector {a,b} if

$Hx(t) = 0$ whenever $x(t) = 0$ and if $a \leq \frac{Hx(t)}{x(t)} \leq b$ for

$x(t) \neq 0$.

(2) H is instantaneously outside the sector {a,b} if

$$a \leq b \text{ and either } \frac{Hx(t)}{x(t)} \leq a \text{ or } \frac{Hx(t)}{x(t)} \geq b \text{ for } x(t) \neq 0.$$

(3) H is instantaneously positive if $x(t)Hx(t) \geq 0.$

A graphical representation of the above conditions is provided by Fig. 4. It is easily seen that if the point (x(t), Hx(t)) always lies in the appropriate shaded region of the plane then the appropriate instantaneous sector condition is satisfied.

Of particular interest here is the memoryless nonlinearity defined below.

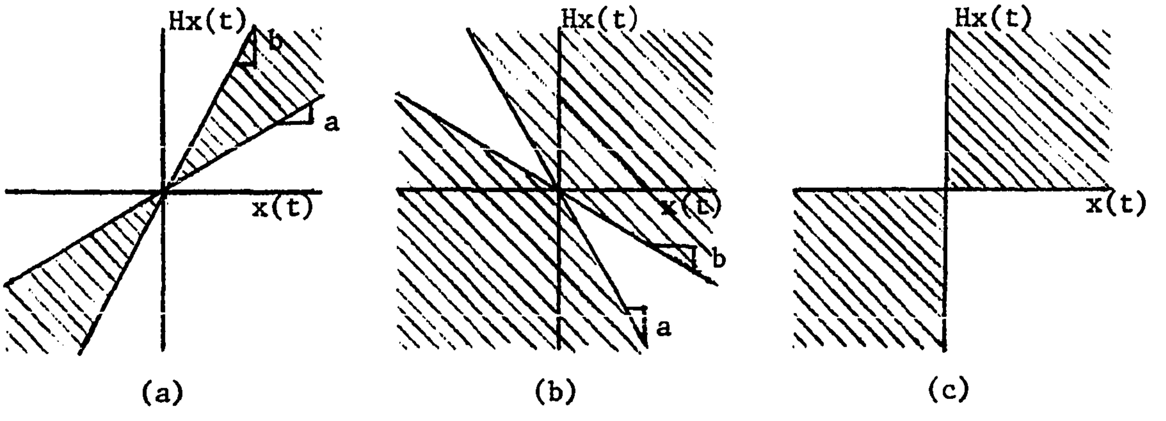


Fig. 4. Interpretation of sector conditions in output versus input plane.

(a) Inside {a,b}.

(b) Outside {a,b}.

(c) Positive.

Definition: A relation H on X_e is memoryless if there exists a real-valued function N such that the equation $Hx(t) = N[x(t)]$ is always satisfied.|

If N is a function of t also, then a time-varying nonlinearity results. Further, a hysterisis nonlinearity can be thought of as corresponding to a multiple-valued N. The above nonlinearities lend themselves particularly well to analysis in terms of instantaneous sector conditions.

The following lemma establishes the usefulness of the instantaneous conditions.

Lemma 2: If the relation H on $L_{2e}[0,\infty)$ is instantaneously inside (outside) the sector $\{a,b\}$, then H is inside (outside) the sector $\{a,b\}$. Also, an instantaneously positive relation is positive. Further, the converse of the above is true for a memoryless relation.|

Incremental counterparts to the instantaneous sector conditions are provided by the following definition.

Definition: Assume H is a relation on $L_{2e}[0,\infty)$. Each of the following statements must be true for all $x,y \epsilon Do(H)$, all $Hx,Hy \epsilon Ra(H)$, and all $t \epsilon T$.

(1) H is instantaneously incrementally inside the sector $\{a,b\}$ if $Hx(t) = Hy(t)$ whenever $x(t) = y(t)$ and if $a \leq \frac{Hx(t)-Hy(t)}{x(t)-y(t)} \leq b$ for $x(t) \neq y(t)$.

(2) H is instantaneously incrementally outside the sector

$\{a,b\}$ if $a \le b$ and either $\frac{Hx(t)-Hy(t)}{x(t)-y(t)} \le a$ or $\frac{Hx(t)-Hy(t)}{x(t)-y(t)}$

$\ge b$ for $x(t) \ne y(t)$.

(3) H is instantaneously incrementally positive if

$[x(t) - y(t)][Hx(t) - Hy(t)] \ge 0.$ |

For a memoryless relation H on $L_{2e}[0,\infty)$, Fig. 5 provides an illustration of (1) and (2) above.

It is easily shown that H is instantaneously incrementally inside the sector $\{a,b\}$ if for each point P of the graph of N the rest of the

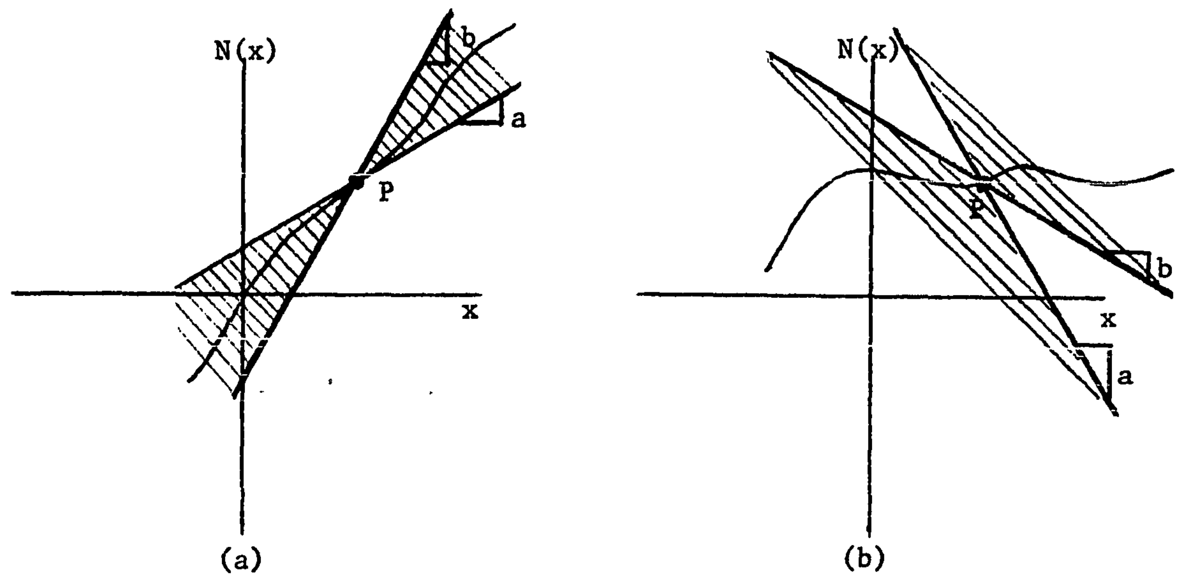


Fig. 5. Incremental sector conditions in output versus input plane.

(a) Incrementally inside $\{a,b\}$.

(b) Incrementally outside $\{a,b\}$.

graph lies in the shaded region of the figure. Clearly if N is differentiable this is equivalent to requiring $a \leq \frac{dN(x)}{dx} \leq b$ for all x. Similarly to the above, H is instantaneously incrementally outside the sector {a,b} if for each point P of the graph of N the rest of the graph lies in the shaded region of the figure. Finally H is instantaneously incrementally positive if N is a nondecreasing function.

The following lemma is the incremental counterpart of Lemma 2.

Lemma 3: If all conditions of Lemma 2 are replaced by their incremental counterparts, then the lemma remains true.|

The interpretations of sector conditions presented here result in a frequency-domain stability condition for a single-loop having an open loop composed of a linear relation in Q and a time-varying nonlinearity. By using Lemmas 1 and 2 in conjunction with Theorem 5, it is easily seen that essentially the same result is obtained as cited earlier due to Sandberg for the n = 1 case. In [16] Zames utilizes this result with a certain transformation to obtain the familiar Papov stability conditions. Further, these results can be extended to L_∞-stability [14].

CHAPTER 4: MAIN RESULTS

New stability results obtained in a functional analysis setting are presented here. Specifically, conditions sufficient to guarantee boundedness and continuity of a multiple-loop nonlinear time-varying system are derived. First a precise mathematical model of a multiple-loop system is given in the form of a particular interconnection of relations. Then a boundedness theorem is presented which involves the interconnection structure and the gains of the relations interconnected. Next a particular transformation of a multiple-loop system is discussed. This leads to a generalization of boundedness results by allowing gain conditions to be replaced by sector conditions. Then a set of conditions are given which guarantee boundedness of a single-loop system. A system satisfying these conditions is referred to as having a margin of boundedness δ. It is found these conditions are useful for the analysis of multiple-loop systems. This chapter is concluded by presentation of continuity results obtained through application of boundedness results to a special system.

System Configuration

A portion of the block diagram of a multiple-loop system is shown in Fig. 6. The purpose of this figure is to indicate a multiple-loop system is an interconnection of relations each having an input composed of a general system input $a_i x$ plus a fixed bias w_i plus a weighted sum of outputs of other relations.

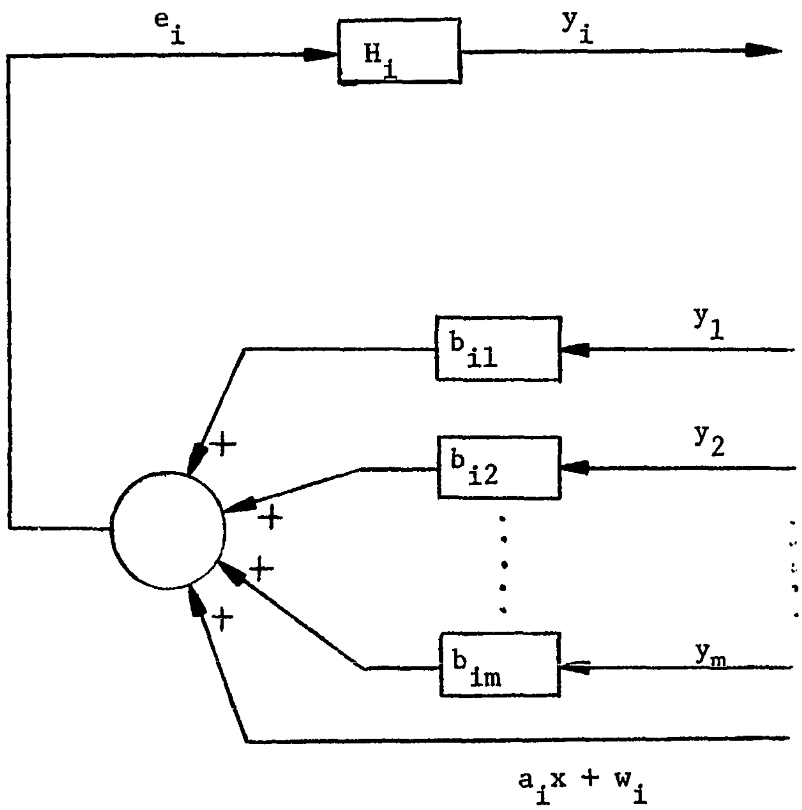


Fig. 6. Multiple-loop system.

In mathematical terms the model of a multiple-loop system is provided by the set of m simultaneous functional equations

$$e_i = a_i x + w_i + \sum_{j=1}^{m} b_{ij} y_j \qquad \text{for } i = 1,2,\ldots,m \qquad (2a)$$

$$y_i = H_i e_i \qquad \text{for } i = 1,2,\ldots,m \qquad (2b)$$

where the following are true:

Each H_i is a relation on X_e.

$x \epsilon X_e$ is the system input.

Each a_i is a constant.

Each w_i is a fixed bias in X.

Each b_{ij} is a constant.

Each $e_i \varepsilon X_e$ is a system output.

Each $y_i \varepsilon X_e$ is a system output.

It should be noted that just as in the case of a single-loop system the bias terms can be used to account for initial condition responses.

For purposes of stability investigations attention is focused on relations which connect the input x with each of the outputs. E_i connects x with e_i, and F_i connects x with y_i. More precisely for $i = 1,2,\ldots,m$

$$E_i = \{(x,e_i):(x,e_i)\varepsilon X_e \times X_e \text{ and there exist } e_j \text{ for all } j \neq i,$$

$$y_j \text{ for all } j, \text{ and } H_j e_j \text{ for all } j \text{ all in } X_e \text{ such}$$

$$\text{that equations (2) are satisfied}\}$$

and

$$F_i = \{(x,y_i):(x,y_i)\varepsilon X_e \times X_e \text{ and there exist } e_j \text{ for all } j,$$

$$y_j \text{ for all } j \neq i, \text{ and } H_j e_j \text{ for all } j \text{ all in } X_e \text{ such}$$

$$\text{that equations (2) are satisfied}\}.$$

A Gain Result

The following theorem gives sufficient conditions for boundedness of a multiple-loop system.

Theorem 7: All relations E_i and F_i associated with the multiple-loop system (2) are bounded if $g(H_i) < \infty$ for all i and the successive principal minors of the matrix

$$I - [|b_{ij}|g(H_j)]$$

are all positive.|

Remark 1: Gain enters into the proof of Theorem 7 only through the inequality $||(H_i x)_t|| \leq g(H_i)||x_t||$. Hence, it might as well be assumed that each H_i is conic with center parameter zero and radius parameter r_i. If X is an inner product space, this is equivalent to H_i being inside the sector $\{-r_i, r_i\}$. In this situation the boundedness condition would be the successive principle minors of $I - [|b_{ij}|r_j]$ are all positive.|

Proof of Theorem 7: It is sufficient to show that each relation E_i is bounded since this implies each relation F_i is also bounded. This follows from

$$||y_i||_e = ||H_i e_i||_e \leq g(H_i)||e_i||_e$$

and the condition that $g(H_i) < \infty$. Clearly F_i is bounded if E_i is bounded.

Now boundedness of E_i is established for each i. Let x, e_i, y_i, and H_ie_i be functions in X_e for each i which satisfy equations (2). Truncate equations (2a) at $t\varepsilon T$ giving for each i

$$e_{it} = a_i x_t + w_{it} + \sum_{j=1}^{m} b_{ij} y_{jt}.$$

Noting all truncated functions lie in X, it is found that for each i

$$||e_{it}|| \leq |a_i| \, ||x_t|| + ||w_{it}|| + \sum_{j=1}^{m} |b_{ij}| \, ||y_{jt}||.$$

From the definition of gain,

$$||e_{it}|| \leq |a_i| \, ||x_t|| + ||w_{it}|| + \sum_{j=1}^{m} |b_{ij}| g(H_j) ||e_{jt}||.$$

Now translate this into matrix notation by using the following definitions:

$$e_t = [||e_{1t}||, ||e_{2t}||, \ldots., ||e_{mt}||]^T,$$

$$h = [|a_1|, |a_2|, \ldots., |a_m|]^T,$$

$$w_t = [||w_{1t}||, ||w_{2t}||, \ldots., ||w_{mt}||]^T.$$

Then for all $t\varepsilon T$

$$e_t \leq h \, ||x_t|| + w_t + [|b_{ij}| g(H_j)] e_t.$$

This gives

$$\{I - [|b_{ij}| g(H_j)]\} e_t \leq h \, ||x_t|| + w_t.$$

Now clearly if $\{I - [|b_{ij}|g(H_j)]\}^{-1}$ exists and has all of its elements nonnegative, then for all $t \in T$

$$e_t \leq \{I - [|b_{ij}|g(H_j)]\}^{-1} \{h||x_t|| + w_t\}.$$

It is shown in Appendix B that the hypotheses of the theorem are sufficient to guarantee this. It follows that there exist constants $f_i \geq 0$ and $k_{ij} \geq 0$ such that for each i and for all $t \in T$

$$||e_{it}|| \leq f_i||x_t|| + \sum_{j=1}^{m} k_{ij}||w_{jt}||.$$

Now assume x belongs to a bounded subset of X_e. Since each $w_i \in X$ and since $||e_{it}||$, $||x_t||$, and $||w_{jt}||$ are all nondecreasing functions of t, letting $t \to \infty$ for each i implies

$$||e_i|| \leq f_i||x|| + \sum_{j=1}^{m} k_{ij}||w_j||.$$

Since the w_j are fixed and since there exists a constant D such that $||x|| < D$, it is clear that each relation E_i is bounded.|

A system composed of a single loop of m relations in cascade is a special case of a multiple-loop system. Theorem 7 provides interesting boundedness conditions for such a system. The B matrix is founded to be of the form

$$B = \begin{bmatrix} 0 & 0 & \cdots\cdots & 0 & b_{1m} \\ b_{21} & 0 & \cdots\cdots & 0 & 0 \\ 0 & b_{32} & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & b_{m,m-1} & 0 \end{bmatrix} .$$

From this it follows that

$$I - [|b_{ij}|g(H_j)] = \begin{bmatrix} 1 & 0 & \cdots & 0 & -|b_{1m}|g(H_m) \\ -|b_{21}|g(H_1) & 1 & \cdots & 0 & 0 \\ 0 & -|b_{32}|g(H_2) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & -|b_{m,m-1}|g(H_{m-1}) & 1 \end{bmatrix} .$$

It is easily found the first m-1 successive principal minors of the above matrix are each unity. The boundedness condition which comes from the last successive principal minor is

$$|b_{1m}|g(H_1)|b_{21}|g(H_2)|b_{32}|g(H_3)\cdots\cdots|b_{m,m-1}|g(H_m) < 1.$$

Hence, it is found even for a single loop of several relations that an open loop gain product less than unity implies boundedness. Certainly Theorem 3 of Chapter 3 is a special case of the above.

It is worth-while to note from the proof of Theorem 7 that specific bounds on system outputs can be found in terms of a bound on the system input x. Hence, if quantitative information is desired the theory is capable of providing it.

For a system found to be bounded from Theorem 7, it is seen the system remaining after removal of any relation H_i is also bounded. To show this assume the hypotheses of Theorem 7 are satisfied. From Theorem 11, stated in Appendix C, it is implied that all principal minors of the matrix $I - [|b_{ij}|g(H_j)]$ are positive. Now removal of the i^{th} relation is equivalent to deletion of the i^{th} row and column of this matrix. But this leaves a matrix which has all principal minors positive. Hence, the system with the i^{th} relation removed is also bounded by Theorem 7.

In certain situations it is desired that stability be retained even if part of the system is disconnected. For such situations Theorem 7 is particularly well adapted. However, it would obviously be useless to try and use this theorem directly in the design of feedback compensation.

Transforming the System

A transformation of a multiple-loop system is discussed here which allows Theorem 7 to find much wider application than at first appears possible. From Remark 1 it is seen if X is an inner product space that boundedness conditions obtained from Theorem 7 require each H_i to be inside a symmetric sector $\{-r_i, r_i\}$. Through a transformation a boundedness theorem can be derived having hypotheses which require each H_i either to be inside or outside a particular sector. Clearly the latter boundedness results encompass a wider variety of situations.

The basic approach employed here is to develop a transformed system of the same form as (2) having a set of solutions which contains the set of solutions of (2). Then boundedness of the transformed system implies boundedness of system (2). It is then found if an appropriate sector condition is satisfied by each H_i that the conditions of Theorem 7 are satisfied for the transformed system. These sector conditions then guarantee boundedness of the set of equations (2).

Now equations (2a) are placed in a matrix format by making the following definitions:

$$e = [e_1, e_2, \ldots., e_m]^T,$$

$$a = [a_1, a_2, \ldots., a_m]^T,$$

$$w = [w_1, w_2, \ldots., w_m]^T,$$

$$y = [y_1, y_2, \ldots., y_m]^T,$$

$$B = [b_{ij}].$$

This results in the following equations:

$$e = ax + w + By, \tag{2a}$$

$$y_i = H_i e_i \quad \text{for } i = 1, 2, \ldots., m. \tag{2b}$$

Now the equations of a multiple-loop system referred to as the transformed system are given. First let A and C be disjoint subsets of the real line such that $A \cup C = \{1, 2, \ldots., m\}$. Then for each

$i \epsilon A \cup C$ pick constants d_i and c_i such that $d_i = 0$ if $i \not\in A$ and $c_i = 0$ if $i \not\in C$. Next define the relation H_i' by

$$H_i' = \begin{Bmatrix} H_i + d_i I & \text{if } i \epsilon A \\ (H_i^{-1} + c_i I)^{-1} & \text{if } i \epsilon C \end{Bmatrix} .$$

Now assume the inverse of the matrix $I + B[\text{diag} d_i]$ exists and make the following definitions:

$$a' = (I + B[\text{diag } d_i])^{-1} a,$$

$$w' = (I + B[\text{diag } d_i])^{-1} w,$$

$$B' = (I + B[\text{diag } d_i])^{-1} (B + [\text{diag } c_i]),$$

$$e' = [e_1', e_2', \ldots, e_m']^T,$$

$$y' = [y_1', y_2', \ldots, y_m']^T.$$

The equations of the transformed system corresponding to the system modeled by equations (2) are then given by the following:

$$e' = a'x + w' + B'y', \qquad (3a)$$

$$y_i' = H_i' e_i' \quad \text{for } i = 1, 2, \ldots, m. \qquad (3b)$$

Clearly these equations are of the same form as equations (2).

Now assume x, e_i, y_i, and $H_i e_i$ are functions in X_e for each i such that equations (2) are satisfied. It is shown now that this solution for equations (2) can be used to find a solution for equations (3). Imagine placing a feedback of $-c_i I$ or a feed-forward of $d_i I$

around each relation H_i in Fig. 6. This is illustrated in Fig. 7. By using standard block diagram manipulations to adjust the interconnection, essentially the same system is retained.

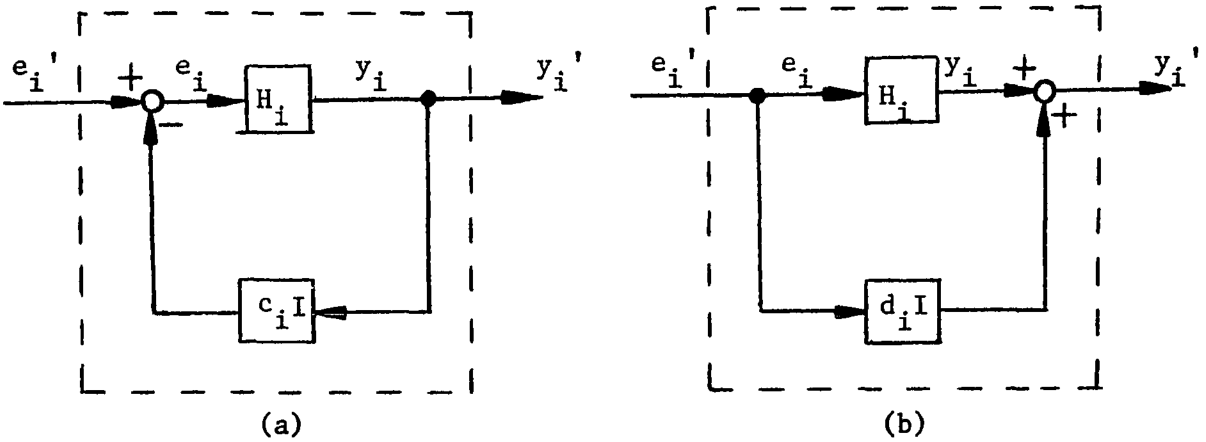


Fig. 7. Feedback and feed-forward around relations of Fig. 6.
(a) $i \varepsilon C$.
(b) $i \varepsilon A$.

Now the primed inputs and outputs of the single-loop systems of Fig. 7 are shown to satisfy equations (3). Specifically, these primed inputs and outputs are defined by the equations $e_i' = e_i + c_i y_i$ and $y_i' = y_i + d_i e_i$ for each i. In a matrix format this becomes

$$\begin{bmatrix} e' \\ y' \end{bmatrix} = \begin{bmatrix} I & [\text{diag } c_i] \\ [\text{diag } d_i] & I \end{bmatrix} \begin{bmatrix} e \\ y \end{bmatrix} .$$

Multiplication by the inverse matrix gives both e and y in terms of e' and y' through the equation

$$\begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} I & -[\text{diag } c_i] \\ -[\text{diag } d_i] & I \end{bmatrix} \begin{bmatrix} e' \\ y' \end{bmatrix} .$$

Substituting for e and y in equation (2a) gives the equation

$$e' - [\text{diag } c_i]y' = ax + w + B(-[\text{diag } d_i]e' + y').$$

After rearranging this equation, it is found

$$(I + B[\text{diag } d_i])e' = ax + w + (B + [\text{diag } c_i])y'.$$

Multiplication by $(I + B[\text{diag } d_i])^{-1}$ gives

$$e' = a'x + w' + B'y'.$$

Hence, (3a) is satisfied.

Now assume $i \varepsilon A$. Then $y_i' = y_i + d_i e_i = H_i e_i + d_i e_i$. Thus, there exists $(H_i + d_i I)e_i$ such that $y_i' = (H_i + d_i I)e_i$. Since $H_i' = H_i + d_i I$ and $e_i' = e_i$, there exists $H_i' e_i'$ such that $y_i' = H_i' e_i'$. For this situation then (3b) is satisfied.

Now assume $i \varepsilon C$. Then $y_i = H_i e_i = H_i(e_i' - c_i y_i)$. This implies there exists $H_i^{-1} y_i$ such that $H_i^{-1} y_i = e_i' - c_i y_i$. This means there exists $(H_i^{-1} + c_i I)y_i$ such that $(H_i^{-1} + c_i I)y_i = e_i'$. But this in turn implies there exists $(H_i^{-1} + c_i I)^{-1} e_i'$ such that $(H_i^{-1} + c_i I)^{-1} e_i' = y_i$. Since $H_i' = (H_i^{-1} + c_i I)^{-1}$ and $y_i' = y_i$, it is then known there exists $H_i' e_i'$ such that $y_i' = H_i' e_i'$. Thus, (3b) is also satisfied in this situation.

Hence, it has been shown that for each solution of equations (2) there is a corresponding solution of equations (3). In this sense the set of solutions of (2) is a subset of the set of solutions of (3). From this it is deduced that boundedness of the multiple-loop transformed system (3) implies boundedness of the multiple-loop system (2).

Using the obvious definitions for the relations E_i' and F_i' associated with the multiple-loop system (3), assume each E_i' and F_i' is bounded. Let x belong to a bounded subset of X_e, and let x, e_i, y_i, and $H_i e_i$ be functions in X_e for each i which satisfy equations (2). There exists a solution of equations (3) with $e_i' = e_i + c_i y_i$ and $y_i' = y_i + d_i e_i$. This leads to the equations $e_i = e_i' - c_i y_i'$ and $y_i = y_i' - d_i e_i'$. But e_i' and y_i' belong to bounded subsets of X_e for each i. Hence, $||e_i|| \leq ||e_i'|| + |c_i|\ ||y_i'||$ and $||y_i|| \leq ||y_i'|| + |d_i|\ ||e_i'||$. From this it is clear each E_i and F_i associated with the multiple-loop system (2) is bounded.

The following example illustrates the transformation through the use of block diagrams.

Example 1: A multiple-loop system comprised of three relations is shown in Fig. 8. The transformed system is represented in Fig. 9 where the primed blocks can be envisioned in terms of the single-loop systems shown.

The constants c_3, d_1, and d_2 are set equal to zero. The B matrix is given by

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix} .$$

This results in

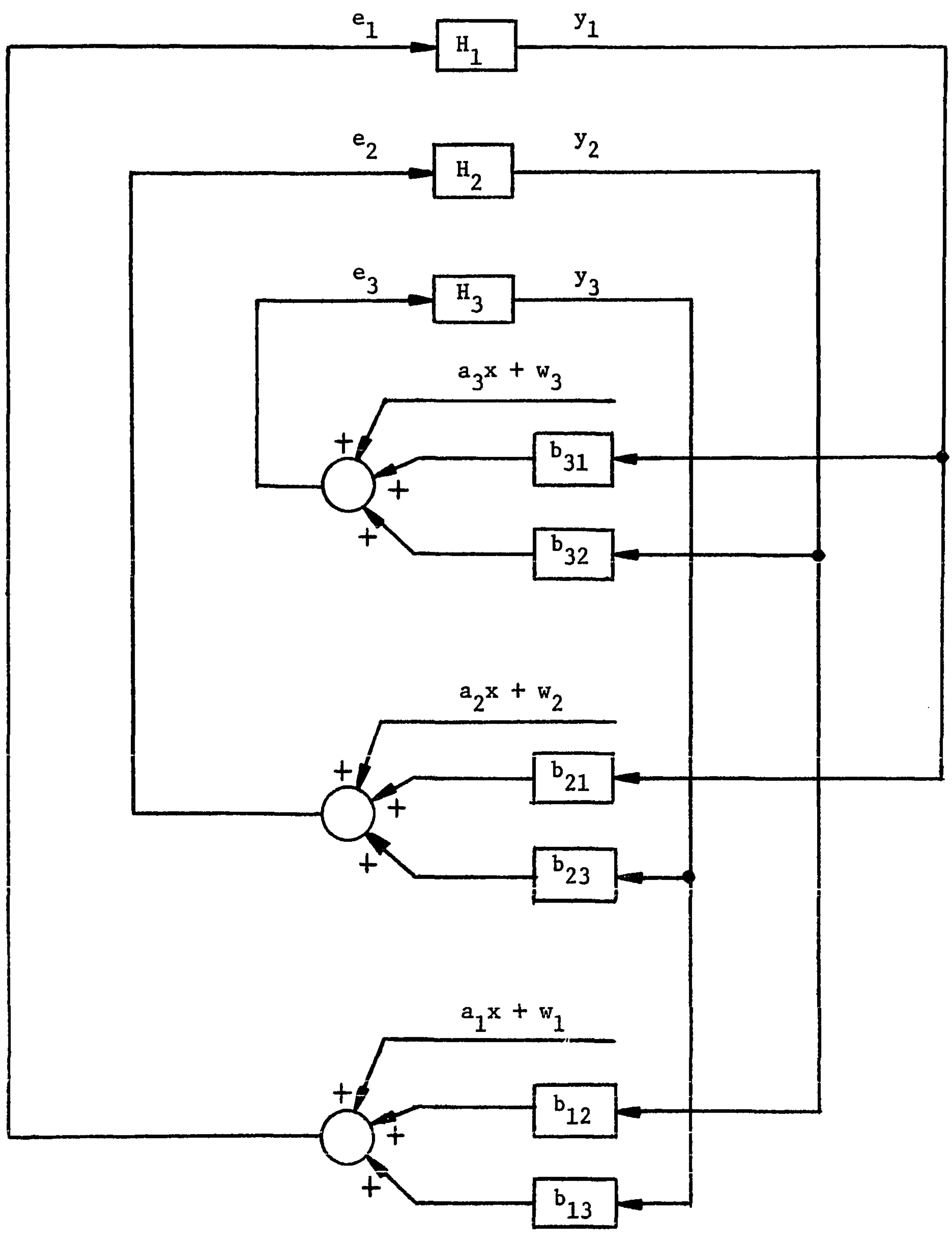


Fig. 8. Multiple-loop system of Example 1.

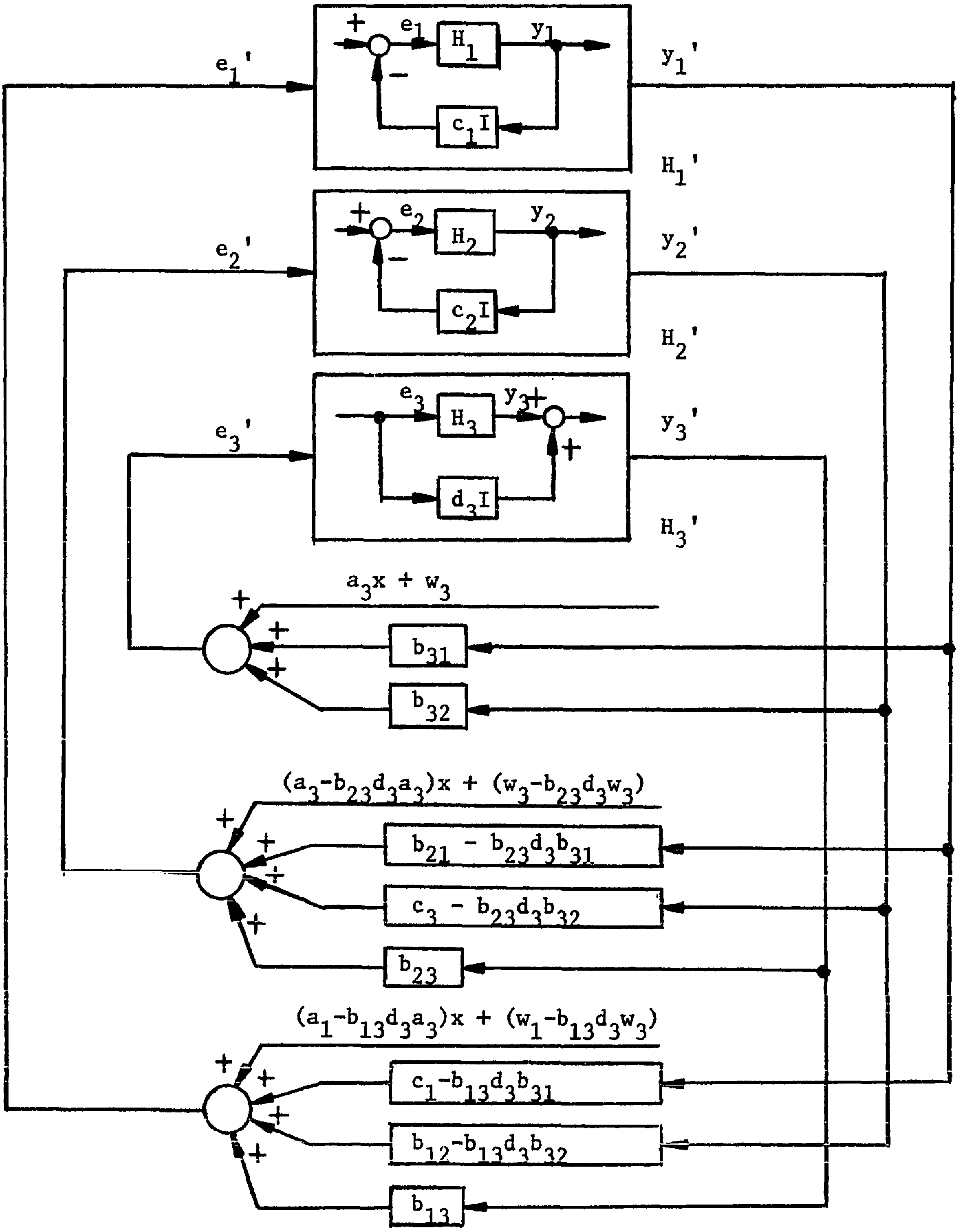


Fig. 9. Transformed system for Example 1.

$$(I + B[\text{diag } d_i])^{-1} = \begin{bmatrix} 1 & 0 & -b_{13}d_3 \\ 0 & 1 & -b_{23}d_3 \\ 0 & 0 & 1 \end{bmatrix} .$$

Hence,

$$a' = \begin{bmatrix} 1 & 0 & -b_{13}d_3 \\ 0 & 1 & -b_{23}d_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1-b_{13}d_3a_3 \\ a_2-b_{23}d_3a_3 \\ a_3 \end{bmatrix} ,$$

$$w' = \begin{bmatrix} w_1-b_{13}d_3w_3 \\ w_2-b_{23}d_3w_3 \\ w_3 \end{bmatrix} ,$$

and

$$B' = \begin{bmatrix} 1 & 0 & -b_{13}d_3 \\ 0 & 1 & -b_{23}d_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & b_{12} & b_{13} \\ b_{21} & c_2 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix} =$$

$$\begin{bmatrix} c_1 - b_{13}d_3b_{31} & b_{12} - b_{13}d_3b_{32} & b_{13} \\ b_{21} - b_{23}d_3b_{31} & c_2 - b_{23}d_3b_{32} & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix} .$$

Also, of course, $H_1' = (H_1^{-1} + c_1I)^{-1}$, $H_2' = (H_2^{-1} + c_2I)^{-1}$, and $H_3' = H_3 + d_3I$. An examination of the two figures shows one can be obtained from the other by standard block diagram manipulations.

A Sector Result

For purposes of convenience slightly different notation is used here when specifying the nature of a conic relation. The incremental counterpart of this notation is obvious.

Definition: A relation H is conic with constants (a,b) if for $a \leq b$ the relation is inside {a,b} and for $a > b$ the relation is outside {b,a}.|

The following theorem guarantees boundedness of a multiple-loop system (2) if certain sector conditions are satisfied. Note X is assumed to be an inner product space.

Theorem 8: Suppose for each i that H_i is conic with constants (a_i, b_i) where $a_i \neq b_i$. Let A and C be disjoint subsets of the real line such that $A \cup C = \{1,2,\ldots,m\}$. Define the constants d_i and c_i by

$$d_i = \begin{cases} -\frac{1}{2}(b_i + a_i) & \text{if } i \in A \\ 0 & \text{if } i \in C \end{cases}$$

and

$$c_i = \begin{cases} 0 & \text{if } i \in A \\ -\left(\frac{b_i + a_i}{2 b_i a_i}\right) & \text{if } i \in C \end{cases}.$$

Let the matrix B' be specified by

$$B' = [b_{ij}'] = (I + B[\text{diag } d_i])^{-1}(B + [\text{diag } c_i])$$

where it is assumed the indicated inverse exists. Further, define $\eta_i > 0$ by

$$\eta_i = \begin{Bmatrix} \frac{1}{2}(b_i - a_i) & \text{if } i\varepsilon A \\ -\left(\frac{2b_i a_i}{b_i - a_i}\right) & \text{if } i\varepsilon C \end{Bmatrix} .$$

Now if the successive principal minors of the matrix

$$I - [|b_{ij}'|\eta_j]$$

are all positive then all relations E_i and F_i associated with the multiple-loop system (2) are bounded.|

Remark 2: The limiting cases for $i\varepsilon C$ of $b_i \to \infty$ or $a_i \to -\infty$ can be rigorously dealt with as explained in Appendix D. For $b_i \to \infty$ the theorem remains true if the hypotheses are changed to read $H_i - a_i I$ is positive and the definitions of c_i and η_i are changed to $c_i = -\frac{1}{2a_i}$ and $\eta_i = -2a_i$. Similarly for $a_i \to -\infty$ the hypotheses are changed to $-H_i + b_i I$ is positive and the definitions of c_i and η_i become $c_i = -\frac{1}{2b_i}$ and $\eta_i = 2b_i$.|

Proof of Theorem 8: Consider the transformed system (3) associated with system (2) for the constants d_i and c_i defined in Theorem 8. It is shown in Appendix C that the relation H_i being conic with constants (a_i, b_i) for each i is sufficient to guarantee H_i' is inside $\{-\eta_i, \eta_i\}$. The equations of the transformed system are of the same form as equations (2). Also, from the definition of w', it is seen that since $w_i \varepsilon X$ for all i that $w_i{}'\varepsilon X$ for all i. Hence, applying Theorem 7, the transformed system is found to be bounded since the successive principal minors of $I - [|b_{ij}'|\eta_j]$ are all positive. But this implies system (2) is bounded.|

The restriction in the theorem that $\eta_i > 0$ imposes interesting constraints on a_i and b_i. For $i\varepsilon A$ it is implied that $a_i < b_i$. If $i\varepsilon C$, then either $a_i < 0$ or $b_i > 0$. Within this constraint all situations are acceptable except for $b_i \geq a_i \geq 0$ and $a_i \leq b_i \leq 0$. This is illustrated in Fig. 10 for the special case of instantaneous sector conditions discussed in Chapter 3.

An illustration of Theorem 8 is provided by considering a multiple-loop system formed from the interconnection of linear time-invariant operators in Q with time-varying nonlinearities. If boundedness conditions are available from Theorem 8, they take the form of conicity requirements on the operators in Q and the time-varying nonlinearities. Under these conicity conditions inputs from a bounded subset of the L_2 space correspond to outputs in bounded subsets of the L_2 space. Assume the i^{th} time-varying nonlinearity is a relation H_i on $L_{2e}[0,\infty)$ which satisfies the equation $H_i x(t) = N_i[x(t),t]$ where N_i is a real-valued function. Then, assuming $i\varepsilon A$, the conicity requirement that H_i be conic with constants (a_i,b_i) is satisfied if the following instantaneous conditions are true:

$$a_i \leq \frac{N_i(x,t)}{x} \leq b_i \text{ for all } x \neq 0 \text{ and all } t\varepsilon[0,\infty),$$

$$N_i(0,t) = 0 \text{ for all } t\varepsilon[0,\infty).$$

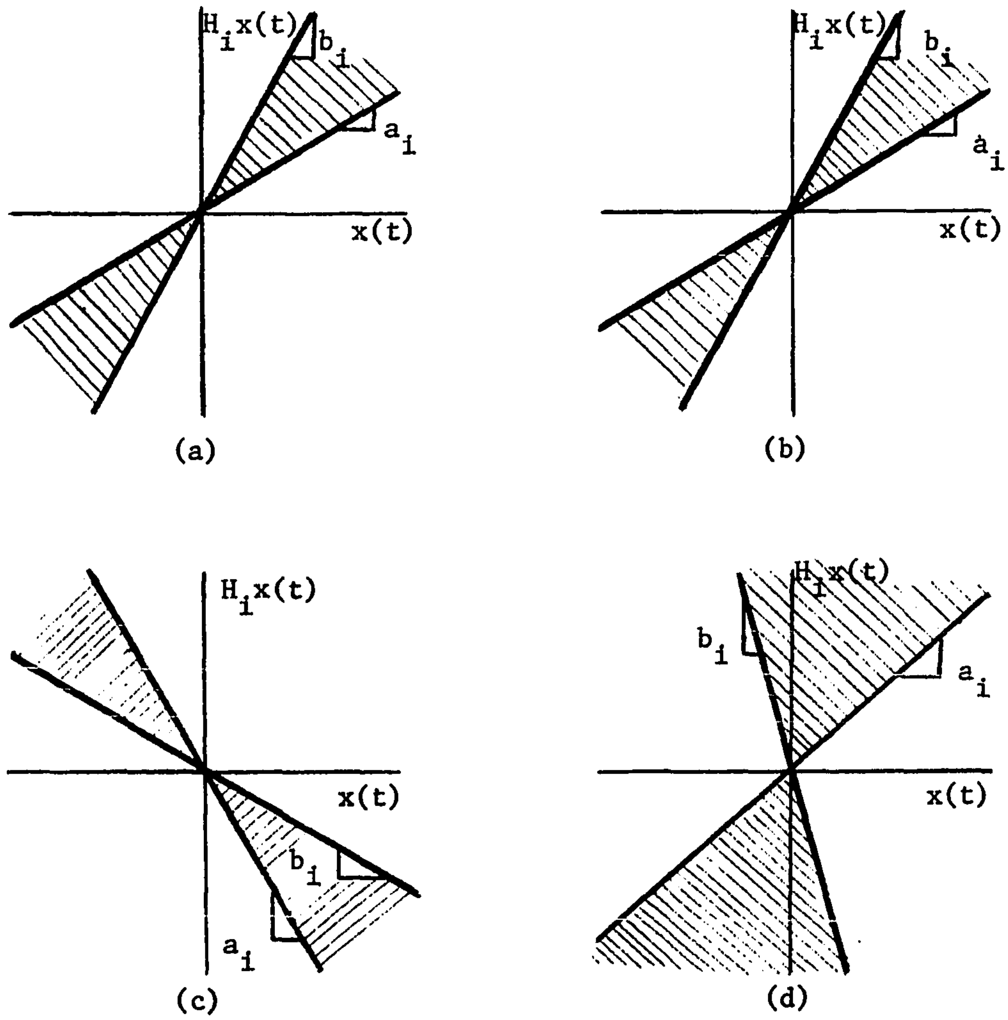


Fig. 10. Instantaneous form of conditions imposed by Theorem 8.

(a) Typical if $i \epsilon A$. (c) Never possible if $i \epsilon C$.

(b) Never possible if $i \epsilon C$. (d) Never possible for any i.

This follows from Lemma 2 of Chapter 3. The conicity requirements on the operators in Q are given an interpretation in the complex plane by Lemma 1 of Chapter 3. The i^{th} linear time-invariant operator is conic with constants (a_i, b_i) if one of the following is true:

(1) For $a_i < b_i$ the Nyquist diagram of the operator lies inside the circle in the complex plane which

intersects the real axis at the points $(a_i,0)$ and $(b_i,0)$.

(2) For $a_i > b_i$ the Nyquist diagram lies outside the circle in the complex plane which intersects the real axis at the points $(b_i,0)$ and $(a_i,0)$. Further, the Nyquist diagram does not encircle the point $(\frac{1}{2}(b_i + a_i),0)$.

The above stability results and those of Theorem 1 due to Sandberg apply to the same general type of system. However, the problem formulation is different since the system dealt with in Theorem 1 is cast in the form of a single loop. Despite this, the only significant difference between the results is in the conditions placed on the linear time-invariant parts. In Theorem 1 a single condition is given involving the supremum over ω of the positive square root of the maximum eigenvalue of a matrix which is a function of ω. Using Theorem 8 results in several conditions involving Nyquist diagrams. This illustrates the fact that a single-loop approach as compared with a multiple-loop approach results in fewer stability conditions which are in general more difficult to verify.

As for Theorem 7, satisfaction of the conditions of Theorem 8 allows specific bounds on system outputs to be found in terms of a bound on the input. This is seen by assuming the conditions of Theorem 8 are satisfied and referring to the discussion of the transformation employed in the proof of the theorem. For each solution of (2) there exists a corresponding primed solution of the appropriate system (3). Assume $x \epsilon X$. Then it is known $||e_i|| \leq ||e_i'|| + |c_i| \, ||y_i'||$. Since H_i' is inside $\{-\eta_i, \eta_i\}$, it is found $||y_i'|| = ||H_i' e_i'|| \leq \eta_i ||e_i'||$. This leads to $||e_i|| \leq (1 + |c_i|\eta_i)||e_i'||$. Now the conditions of Theorem 7 are satisfied for system (3). Thus, from the latter portion of the proof of Theorem 7, it is seen constants $f_i' \geq 0$ and $k_{ij}' \geq 0$ can be calculated such that

$$||e_i|| \leq (1 + |c_i|\eta_i)(f_i'||x|| + \sum_{j=1}^{m} k_{ij}'||w_j'||).$$

Similarly, it is found a bound on $||y_i||$ can be found in terms of $||x||$.

The theory is found to be capable of providing a feeling for the "degree of stability" possessed by a system. Assuming first that the conditions of Theorem 8 are satisfied, it is clear that making restrictions on system parameters more stringent results in tighter bounds on system responses. In this sense the margin by which boundedness conditions are satisfied is a measure of "how stable" a system is.

A Margin of Boundedness δ

The following defines a condition on a single-loop system which is later found to be helpful in the application of Theorem 8 to multiple-loop systems.

Definition: The single-loop system (1) possessing open-loop relations H_1 and H_2 has a margin of boundedness δ if one of the following is true for some $0 < \delta < 1$:

Case 1a: H_1 is conic with constants (a,b) where $b < a < 0$,

and $-H_2$ is inside the sector

$\{-\frac{1}{b} - \delta(\frac{b-a}{2ba}), -\frac{1}{a} + \delta(\frac{b-a}{2ba})\}$.

Case 1b: H_1 is conic with constants (a,b) where $a < 0$ and

$b > 0$. Further, $-H_2$ is inside the sector

$\{-\frac{1}{b} - \delta(\frac{b-a}{2ba}), -\frac{1}{a} + \delta(\frac{b-a}{2ba})\}$.

Case 2: $H_1 - aI$ is positive where $a < 0$, and $-H_2$ is inside

the sector $\{-\delta(\frac{1}{2a}), -\frac{1}{a} + \delta(\frac{1}{2a})\}$.|

The above definition is given an instantaneous interpretation by Fig. 11 for relations on $L_{2e}[0,\infty)$. If each of the points $(x(t), H_1x(t))$ and $(x(t), -H_2x(t))$ always lie in the appropriate shaded regions in the figure, then the single-loop system has a margin of boundedness δ. Actually, from the figure, it is seen that Case 2 can be obtained from Case 1a in the limit as $b \to -\infty$ or from Case 1b in the limit as $b \to \infty$.

The motivation for the above definition is found to be that a single-loop system having a margin of boundedness δ satisfies the conditions of Theorem 8 within that margin. This is shown by noting for Cases 1a and 1b that H_1 is conic with constants (a_1,b_1) and H_2 is conic with constants (a_2,b_2) where $a_1 = a$, $b_1 = b$, $a_2 = \frac{1}{a} - \delta(\frac{b-a}{2ba})$, and $b_2 = \frac{1}{b} + \delta(\frac{b-a}{2ba})$. The conicity of H_2 is inferred from the conicity of $-H_2$ by property (2) in Appendix D. In Case 2, $H_1 - a_1I$ is positive and H_2 is conic with constants (a_2,b_2) where $a_1 = a$, $a_2 = \frac{1}{a} - \delta(\frac{1}{2a})$, and

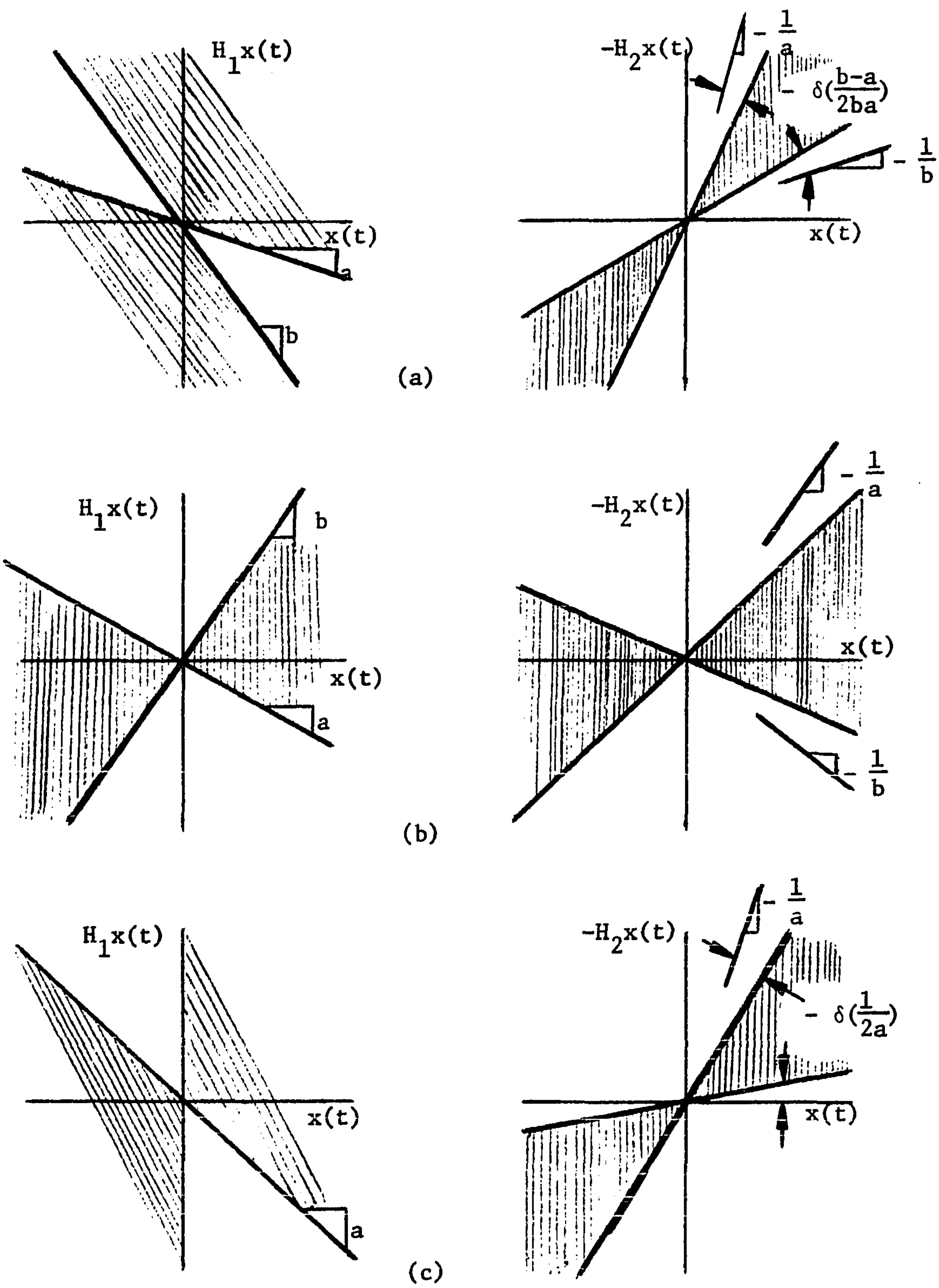


Fig. 11. Instantaneous conditions for a margin of boundedness δ. (a) Case 1a: $b < a < 0$. (b) Case 1b: $b > 0$ and $a < 0$. (c) Case 2: $a < 0$.

$b_2 = \delta(\frac{1}{2a})$. Now it is shown for each case that the conditions of Theorem 8 and Remark 2 are satisfied. Let A = {2} and C = {1}. The matrix B is found to be

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then B' is easily found to be

$$B' = \begin{bmatrix} c_1-d_2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now for Cases 1a and 1b

$$c_1 - d_2 = -(\frac{b_1+a_1}{2b_1a_1}) + \frac{1}{2}(b_2+a_2) = -(\frac{b+a}{2ba}) + \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) = 0.$$

Further, for Case 2

$$c_1 - d_2 = -\frac{1}{2a_1} + \frac{1}{2}(b_2+a_2) = -\frac{1}{2a} + \frac{1}{2}(\frac{1}{a}) = 0.$$

Hence, B' = B indicating the feedback in the transformation is canceled by the feed-forward. Now it is found that

$$I - [|b_{ij}'|\eta_j] = \begin{bmatrix} 1 & -\eta_2 \\ -\eta_1 & 1 \end{bmatrix}$$

so the single-loop system is bounded if

$$1 - \eta_1\eta_2 > 0.$$

For Cases 1a and 1b

$$\eta_1\eta_2 = -\left(\frac{2b_1a_1}{b_1-a_1}\right)[\tfrac{1}{2}(b_2-a_2)] = -\left(\frac{ba}{b-a}\right)[\frac{1}{b} - \frac{1}{a} + 2\delta(\frac{b-a}{2ba})] = 1 - \delta.$$

Further, for Case 2

$$\eta_1\eta_2 = -2a_1[\tfrac{1}{2}(b_2-a_2)] = -a[-\frac{1}{a} + 2\delta(\frac{1}{2a})] = 1 - \delta.$$

Therefore, in each case

$$1 - \eta_1\eta_2 = \delta > 0.$$

Hence, the boundedness condition is satisfied within a margin δ implying the single-loop system is bounded.

The problem which arises concerning use of Theorem 7 for the design of feedback compensation does not occur for Theorem 8. This is easily seen by considering the single-loop system with a margin of boundedness δ. Assume the conditions of Case 1a are satisfied. The system is then bounded by Theorem 8, but removal of the relation H_2 does not leave a system which is required to have a finite gain.

It can be shown by using the idea of a margin of boundedness δ that new results presented here specialize to results presented by Zames [15]. Specifically, it is found that Theorem 5 of Chapter 3 is obtained by applying Theorem 8 of this chapter to a single-loop system. Suppose the conditions of Theorem 5 are satisfied for some $\gamma > 0$. The case $\gamma = 0$ need not be considered since this is shown by Zames to be a special case of $\gamma > 0$. Now compare each case of Theorem 5 with the corresponding case in the definition of a margin of boundedness δ. Clearly a δ can be found so the system being examined has a margin of boundedness δ. But this implies the system is bounded from Theorem 8. Hence, Theorem 8 can be utilized to prove Theorem 5.

Now two examples are given which indicate how the interconnection structure influences the final form of the stability conditions. In order that this influence might be most easily discerned, Theorem 7 is utilized rather than the more complicated Theorem 8. Then it is discussed how the form of the interconnection can be used as a guide for the application of Theorem 8. In this connection it is found the idea of a margin of boundedness δ is quite useful.

Example 2: Consider the multiple-loop system represented by the block diagram of Fig. 12 where H_1, H_2, and H_3 are all relation on X_e of finite gain.

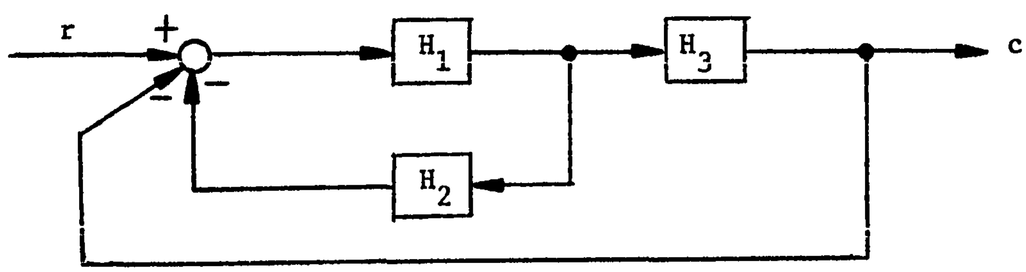


Fig. 12. Block diagram for Example 2.

First the B matrix is found to be

$$B = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

This gives

$$I - [|b_{ij}|g(H_j)] = \begin{bmatrix} 1 & -g(H_2) & -g(H_3) \\ -g(H_1) & 1 & 0 \\ -g(H_1) & 0 & 1 \end{bmatrix} .$$

Then calculating the successive principal minors leads to the boundedness conditions:

$$1 - g(H_1)g(H_2) > 0,$$

$$1 - g(H_1)g(H_2) - g(H_1)g(H_3) > 0.$$

Since gains are positive, the first condition can be eliminated because it is implied by the second. The second condition also implies

$$1 - g(H_1)g(H_3) > 0.$$

From this it is clear that if the multiple-loop system of Fig. 12 is bounded from Theorem 7, then each subloop of this system has an open-loop gain less than unity. Hence, from Theorem 7 each subloop is bounded. However, the converse of this is not true unless each subloop satisfies the conditions of Theorem 7 within a certain margin. Define δ_1 and δ_2 by

$$1 - g(H_1)g(H_2) = \delta_1$$

and

$$1 - g(H_1)g(H_3) = \delta_2.$$

Then if $\delta_1 + \delta_2 > 1$ the multiple-loop system is bounded.|

Example 3: The system examined here is shown in Fig. 13 where H_1, H_2, and H_3 are each relations on X_e of finite gain. The constants k_1 and k_2 are positive. The B matrix is easily found to be

$$B = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & k_1 \\ 1 & k_2 & 0 \end{bmatrix}.$$

Then it is seen that

$$I - [|b_{ij}|g(H_j)] = \begin{bmatrix} 1 & -g(H_2) & -g(H_3) \\ -g(H_1) & 1 & -k_1 g(H_3) \\ -g(H_1) & -k_2 g(H_2) & 1 \end{bmatrix}.$$

Calculation of the successive principal minors of the above matrix gives the boundedness conditions:

$$1 - g(H_1)g(H_2) > 0,$$

$$1 - g(H_1)g(H_2) - g(H_1)g(H_3) - k_1k_2g(H_3)g(H_2) - k_1g(H_1)g(H_2)g(H_3)$$
$$- k_2g(H_1)g(H_2)g(H_3) > 0.$$

The first condition can be eliminated since it is implied by the second. Actually the second condition implies each of the following:

$$1 - g(H_1)g(H_2) > 0, \qquad (4a)$$

$$1 - g(H_1)g(H_3) > 0, \qquad (4b)$$

$$1 - k_1k_2g(H_3)g(H_2) > 0, \qquad (4c)$$

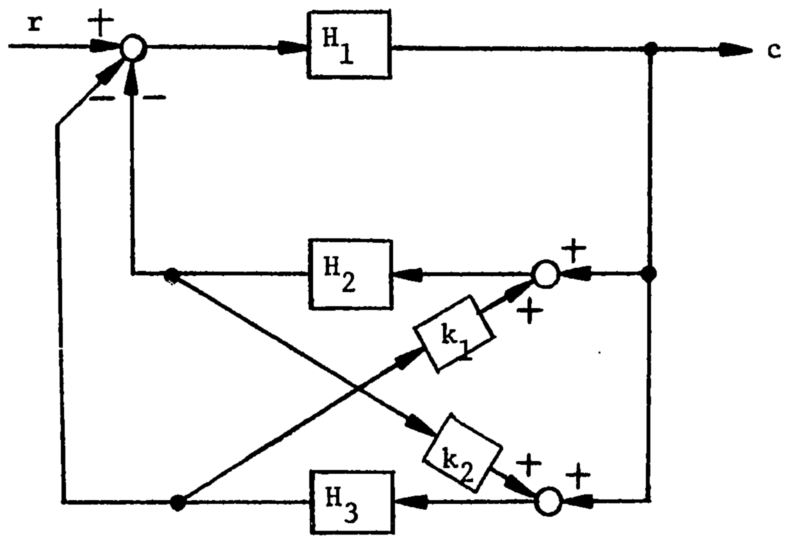


Fig. 13. Block diagram for Example 3.

$$1 - k_1 g(H_1) g(H_2) g(H_3) > 0, \qquad (4d)$$

$$1 - k_2 g(H_1) g(H_2) g(H_3) > 0. \qquad (4e)$$

Hence, it is clear that if the multiple-loop system of Fig. 13 is bounded from Theorem 7, then each subloop of the system has open-loop gain less than unity. Thus, from Theorem 7 each subloop is bounded. Again the converse of this is not true unless the subloops each satisfy the conditions of Theorem 7 within a certain margin. If the right hand side of each inequality in (4) is replaced with a positive number such that the sum of these numbers is greater than four, then the multiple-loop system of Fig. 13 is bounded.|

The purpose of the above two examples is to indicate, that as far as Theorem 7 is concerned, boundedness conditions for a multiple-loop system can often be stated in terms of margins by which subloops satisfy boundedness conditions. This information can be used to guide application of Theorem 8 since this theorem actually involves application of Theorem 7 to a transformed system. It can often be seen how boundedness conditions on subloops of the transformed system reflect back into the original system. Here the condition that a subloop of the original system has a margin of boundedness δ becomes useful. This is due to the fact that satisfaction of this condition implies a corresponding subloop of the transformed system satisfies a boundedness condition within a margin δ. Boundedness conditions for the original system then involve δ. This allows an organized approach and often reveals tradeoffs in conditions.

Continuity

By devising a system relating changes in inputs to changes in outputs, continuity results can be obtained almost directly from boundedness results. This approach produces the following theorem.

Theorem 9: If the conditions of Theorem 8 are replaced by their incremental counterparts, then all relations E_i and F_i associated with the multiple-loop system (2) are continuous.|

Remark 3: The limiting cases of $b_i \to \infty$ and $a_i \to -\infty$ for $i \in C$ are handled in the same manner as for Theorem 8 but by requiring incremental conditions to be satisfied.|

Proof: First a system relating changes in inputs to changes in outputs is presented. Define the relation G_i on X_e by

$$G_i = \{(e,y)\text{:there exist } w \text{ and } v \text{ in } Do(H_i) \text{ such that } e = (w-v) \text{ and there exists } H_i w \text{ and } H_i v \text{ such that } y = H_i w - H_i v\}.$$

G_i then relates changes in the input of H_i to changes in its output. Consider the following system:

$$\hat{e}_i = a_i\hat{x} + \sum_{j=1}^{m} b_{ij}\hat{y}_j \quad \text{for } i = 1, 2, \ldots, m, \tag{5a}$$

$$\hat{y}_i = G_i\hat{e}_i \quad \text{for } i = 1, 2, \ldots, m. \tag{5b}$$

Let x_1, e_{i1}, y_{i1}, and H_ie_{i1} be functions in X_e satisfying (2) for each i. Further, let x_2, e_{i2}, y_{i2}, and H_ie_{i2} be functions in X_e also satisfying (2) for each i. Now for $\hat{x} = x_1 - x_2$, $\hat{e}_i = e_{i1} - e_{i2}$, and $\hat{y}_i = y_{i1} - y_{i2}$ there exists $G_i\hat{e}_i$ in X_e such that equations (5) are satisfied. This is easily seen by subtracting the equations corresponding to the input x_2 from those corresponding to the input x_1. The w_i are eliminated since they are fixed regardless of the input. Hence, equations (5) relate changes in the input $\hat{x}$ to changes in the outputs $\hat{e}_i$ and $\hat{y}_i$ through the relations $\hat{E}_i$ and $\hat{F}_i$.

Assume the conditions of Theorem 9 are satisfied. H_i is incrementally conic with constants (a_i,b_i). This implies G_i is conic with constants (a_i,b_i). To see this let $e\varepsilon Do(G_i)$, $G_ie\varepsilon Ra(G_i)$, and $t\varepsilon T$. Then by the definition of G_i there exists a w and v in $Do(H_i)$ such that $e = w - v$ and there exists $H_i w$ and $H_i v$ such that $G_i e = H_i w - H_i v$. Therefore,

$$<(G_i e)_t - a_i e_t, (G_i e)_t - b_i e_t> =$$

$$<(H_i w - H_i v)_t - a_i (w-v)_t, \ (H_i w - H_i v)_t - b_i (w-v)_t>.$$

Hence, incremental conicity requirements on H_i imply corresponding conicity requirements on G_i.

Applying Theorem 8 to system (5), it is seen that $\hat{E}_i$ and $\hat{F}_i$ are bounded for all i. This means bounded changes in the input produce only bounded changes in the output. This is close to what is desired. To actually obtain continuity, the proofs of Theorems 8 and 7 must be examined. The proof of Theorem 8 involves showning an appropriate transformed system (3) is bounded from Theorem 7. Denote inputs and outputs for the transformed system corresponding to (5) by $\hat{e}_i'$ and $\hat{y}_i'$. Since the w_i in equations (5) are zero, the equation for w' in equations (3) indicates w_i' in the transformed system is zero. Hence, from the last portion of the proof of Theorem 7, it is seen there exists $f_i \geq 0$ such that for each i

$$||\hat{e}_i'|| \leq f_i||\hat{x}|| \text{ and } ||\hat{y}_i'|| \leq \eta_i f_i||\hat{x}||$$

for any solution of the transformed system with $\hat{x} \varepsilon X$.

It was shown when discussing the transformation that for each solution of (5) there is a solution of the transformed system such that for each i

$$\hat{e}_i = \hat{e}_i' - c_i\hat{y}_i' \text{ and } \hat{y}_i = \hat{y}_i' - d_i\hat{e}_i'.$$

Thus, for each solution of (5) with $\hat{x}\varepsilon X$ it is seen

$$||\hat{e}_i|| \leq (f_i + |c_i|\eta_i f_i)||\hat{x}|| \text{ and } ||\hat{y}_i|| \leq (\eta_i f_i + |d_i|f_i)||\hat{x}||.$$

Now returning to the beginning of the proof to the solutions corresponding to inputs x_1 and x_2, it is seen that for $x_1 - x_2 \varepsilon X$.

$$||e_{i1}-e_{i2}|| \leq (f_i + |c_i|\eta_i f_i)||x_1-x_2||$$

and

$$||y_{i1}-y_{i2}|| \leq (\eta_i f_i + |d_i|f_i)||x_1-x_2||.$$

Pick $\varepsilon > 0$. Let $\delta_i = \dfrac{\varepsilon}{f_i + |c_i|\eta_i f_i}$. Then $||x_1-x_2|| < \delta_i$ implies $||E_i x_1 - E_i x_2|| = ||e_{i1}-e_{i2}|| \leq (f_i + |c_i|\eta_i f_i)||x_1-x_2|| < \varepsilon$. Hence, E_i is continuous for each i. Similarly all F_i are continuous.|

Now consider a multiple-loop system formed from the interconnection of linear time-invariant operators in Q with memoryless nonlinearities. If Theorem 9 yields continuity conditions, they take the form of incremental conicity requirements. Then satisfaction of these requirements implies inputs arbitrarily close to each other in the L_2 space correspond to outputs arbitrarily close to each other in the L_2 space. Boundedness conditions were discussed earlier for a similar system in connection with Theorem 8. It is found the incremental conicity requirements on the operators in Q have the same interpretation here as earlier. However, the requirements on the nonlinearities are interpreted differently. Assume the i^{th} memoryless nonlinearity is a relation H_i on $L_{2e}[0,\infty)$ which satisfies the equation $H_i x(t) = N_i[x(t)]$ where N_i is a real-valued differentiable function. Then, assuming $i \varepsilon A$, the requirement that H_i be incrementally conic with constants (a_i, b_i) is satisfied if $a_i \leq \frac{dN_i(x)}{dx} \leq b_i$ for all x. This is true by Lemma 3 of Chapter 3.

From the latter portion of the proof of Theorem 9, it is clear satisfaction of the hypotheses of the theorem actually implies more than continuity. The theory is capable of providing quantitative information in the form of specific bounds on deviations in system outputs in terms of deviation in the system input. Under initial satisfaction of the conditions of Theorem 9, a further tightening of restrictions on system parameters clearly results in tighter bounds on deviations in outputs. Thus, a similar situation exists as for Theorem 8 in that the margin within which conditions of Theorem 9 are satisfied can be viewed as a measure of "how stable" a system is.

CHAPTER 5: APPLICATIONS

In this chapter the stability of several systems is investigated through the use of Theorems 8 and 9. For each system examined, it is found the form the interconnection structure takes is a helpful guide for application of the theory. This is reflected by the fact that in each instance general stability conditions always require certain subloops to have a margin of boundedness δ.

For each system investigated, boundedness and continuity are interpreted in terms of the L_2 norm. This is done because analysis in terms of this norm allows results to be obtained in the most direct manner. This permits the emphasis to be placed on changes in the theory required to go from a single-loop to a multiple-loop system. Several extensions of the single-loop theory have been made which are interesting to examine relative to the multiple-loop theory. For instance, Zames [14] presents a theorem for L_∞-boundedness which is comparable with Theorem 8 and has a frequency-domain interpretation for a single-loop with a linear part and a nonlinearity.

Due to the fact that the L_2 norm is used exclusively, it must be assumed in all cases that all system inputs and outputs are square integrable over any finite time-interval. From an engineering viewpoint this is a trivial restriction since it is almost always true for any physical system of interest.

Results presented in this chapter are relevant to the Lyapunov type of stability as well as the type of stability defined in a functional analysis setting. This seems reasonable from consideration of a dynamical system for which a bounded set of inputs leads to a bounded set of outputs in the sense of the L_2 norm. Roughly speaking, a zero input then corresponds to an output which becomes small in the remote future regardless of initial conditions. But this is close to the idea of asymptotic stability. Willems [13] makes this more precise by proving that global asymptotic stability in the sense of Lyapunov results if the state space is accessible and observable in some sense.

The first system considered here is a particular interconnection of three specific linear time-invariant systems with three specific time-varying nonlinearities. This is followed by examination of a network possessing passive components. Finally, a particular nonlinear time-varying differential equation involving time delay is analyzed.

Example I: Consider the multiple-loop system shown in Fig. 14. Application of Theorem 8 shows this system is bounded in the sense of the L_2 norm.

First a more detailed description of the system illustrated in the block diagram is given. Clearly this system is an interconnection of three linear time-invariant systems, a time-varying gain, a piecewise linear nonlinearity, and a hysterisis nonlinearity. Let h(t) be the inverse Laplace transform of $\frac{s+20}{(s+1)(s+2)}$. Then the block labeled $\frac{s+20}{(s+1)(s+2)}$ represents a system having input u and output v which satisfy the integral equation

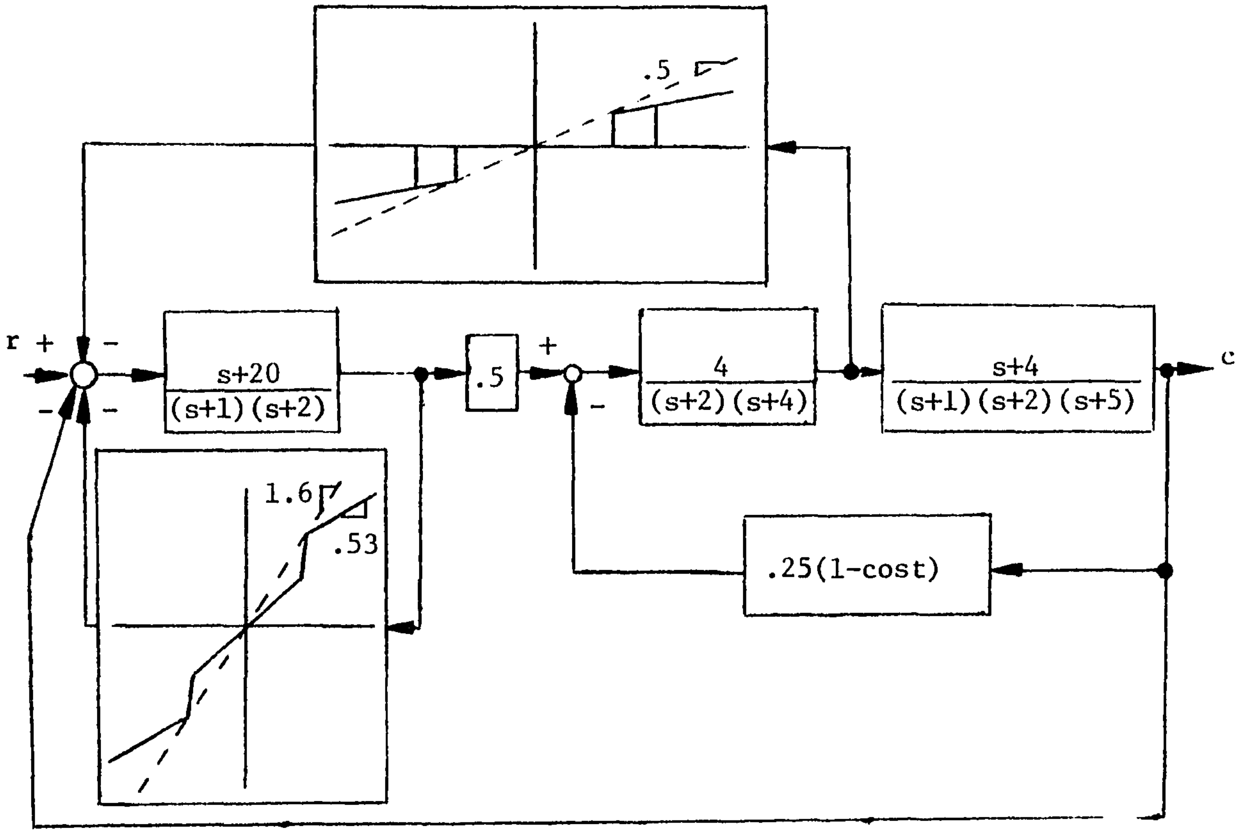


Fig. 14. Multiple-loop system of Example 1.

$$v(t) = z(t) + \int_0^t h(t-\tau)u(\tau)d\tau$$

for $t \geq 0$ where z(t) accounts for initial conditions. It is assumed z(t) lies in $L_2\,[0,\infty)$. The blocks labeled $\frac{4}{(s+2)(s+4)}$ and $\frac{s+4}{(s+1)(s+2)(s+5)}$ represent systems modeled by similar integral equations having initial condition responses in $L_2[0,\infty)$. The assumption is made that each block in Fig. 14 has inputs and outputs which are square integrable over any finite time-interval.

Now consider any input or output of any block in Fig. 14. Boundedness implies a bound on the "size" of this input or output can be given in terms of a bound on the "size" of the input r. For example, consider the output c. Now for each D and each set of initial conditions a C can be calculated so that when a solution exists

$$\int_0^\infty r(t)^2 dt < D$$

implies

$$\int_0^\infty c(t)^2 dt < C.$$

This means, roughly speaking, that inputs which become small rapidly enough in the remote future lead to outputs which become small in the remote future regardless of initial conditions.

The following modification of the system in Fig. 14 is shown to be continuous by Theorem 9. Imagine replacing each of the nonlinearities of the system in Fig. 14 with a slope restricted nonlinearity. Specifically, replace the graphs of the piecewise linear nonlinearity and the hysterisis nonlinearity with, respectively, the real-valued differentiable functions N_1 and N_2. Assume the inequalities $.53 \leq \frac{dN_1(x)}{dx} \leq 1.6$ and $0 \leq \frac{dN_2(x)}{dx} \leq .5$ are satisfied for all x.

Now consider any input or output of any block in the modified system of Fig. 14. Continuity implies a bound on the "size" of the deviation in this input or output can be found in terms of a bound on the "size" of the deviation in the input. Being more specific, consider the output c. Let r and r' be two general system inputs for which under identical initial conditions the, respective, outputs c and c' exist. Then if

$$\int_0^\infty [r(t) - r'(t)]^2 dt < \infty$$

a constant K can be calculated which is independent of initial conditions and for which

$$\int_0^\infty [c(t) - c'(t)]^2 dt \leq K \int_0^\infty [r(t) - r'(t)]^2 dt.$$

Loosely speaking, this means inputs which become close rapidly enough in the remote future lead under identical initial conditions to outputs which become close in the remote future. The fact that the system is continuous also means the jump phenonenon cannot be displayed.

General stability results which specialize to those given above can be obtained for the multiple-loop system of Fig. 15. These results are found from application of Theorems 8 and 9 and are cast in a form which reveal tradeoffs.

First consider in more detail the system depicted by Fig. 15. The blocks labeled H_1, H_2, and H_3 are members of the class of linear time-invariant operators Q. Zero-input responses of the systems modeled by these operators are accounted for by the functions z_1, z_2, and z_3 which are all assumed to be in $L_2[0,\infty)$. The blocks labeled H_4, H_5, and H_6 represent time-varying nonlinearities which are modeled by relations on $L_{2e}[0,\infty)$. The constant k is assumed to be nonnegative. This multiple-loop system clearly has equations of the form of (2)

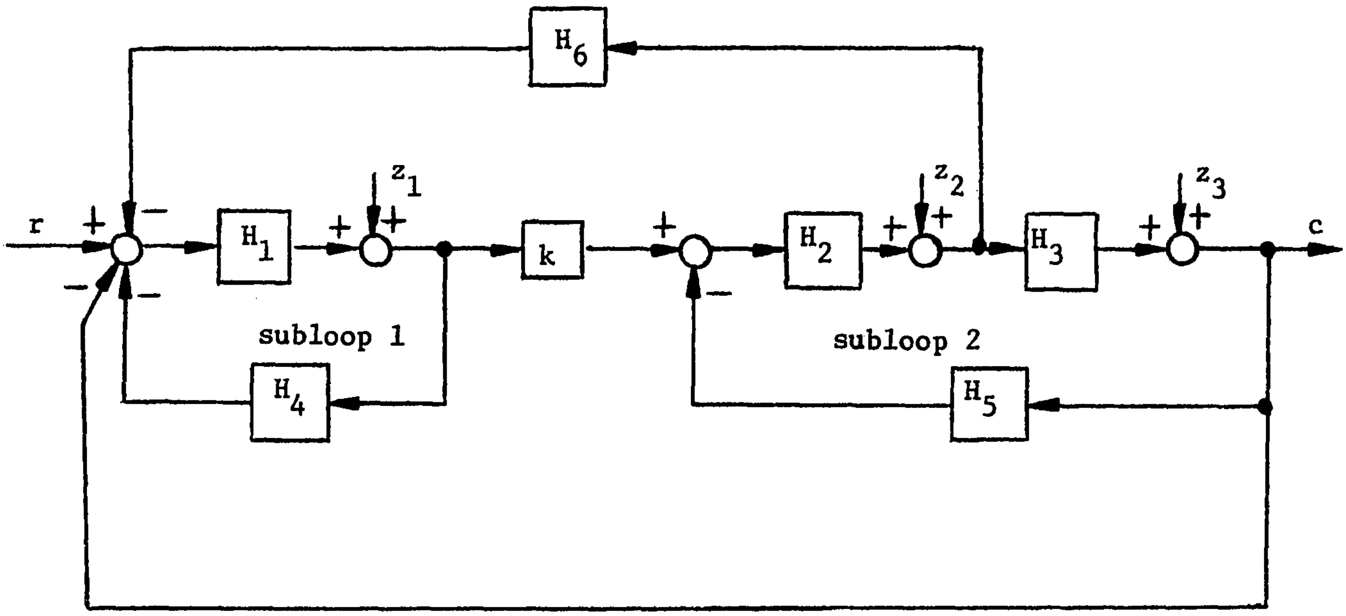


Fig. 15. Multiple-loop system.

where the input r corresponds to x and the w_i functions are linear combinations of z_1, z_2, and z_3.

Use of Theorem 8 yields the following boundedness conditions. Assume the single-loop system possessing open-loop relations H_1 and $-H_4$ has a margin of boundedness δ where H_1 is conic with constants (a_1,b_1). In other words, assume subloop 1 has a margin of boundedness δ. This means H_1 is conic with constants (a_1,b_1) and H_4 is inside $\{-\frac{1}{b_1} - \delta(\frac{b_1-a_1}{2b_1a_1}), -\frac{1}{a_1} + \delta(\frac{b_1-a_1}{2b_1a_1})\}$ for some $0 < \delta < 1$ where either $b_1 < a_1 < 0$ or $a_1 < 0$ anb $b_1 > 0$. Further, assume H_2 is inside $\{-b_2,b_2\}$, H_3 is inside $\{-b_3,b_3\}$, H_5 is inside $\{0,b_5\}$, and H_6 is inside $\{0,b_6\}$ where b_2, b_3, b_5, and b_6 are all positive. Then the system of Fig. 15 is bounded if

$$\delta(1 - b_2b_3b_5) + kb_2(b_3 + b_6)\,\frac{2b_1a_1}{b_1-a_1} > 0.$$

It is interesting to observe that a necessary condition for satisfaction of the above inequality is $b_2b_3b_5 < 1$. Further, if this condition is satisfied a $k \geq 0$ can always be found for which boundedness is guaranteed. This is particularly interesting since from Theorem 7 the condition $b_2b_3b_5 < 1$ guarantees boundedness of subloop 2 comprised of relations H_2, H_3, and H_5. Apparently if subloops 1 and 2 satisfy boundedness conditions and the coupling k between them is weak enough, then the entire multiple-loop system is bounded.

Now the boundedness conditions cited above for the system of Fig. 15 are employed to show the system of Fig. 14 is bounded. Making a comparison of the two figures it is seen the block labeled H_1 corresponds with the block labeled $\frac{s+20}{(s+1)(s+2)}$. Now the block in Fig. 14 is stated above to be modeled by an integral equation having Laplace transform $\frac{s+20}{(s+1)(s+2)}$. Since the poles of this transform lie strictly in the left half plane, this system belongs to the class of linear operators Q. Further, the initial condition response belongs to $L_2[0,\infty)$. Similarly, the blocks labeled H_2 and H_3 correspond, respectively, with the blocks labeled $\frac{4}{(s+2)(s+4)}$ and $\frac{s+4}{(s+1)(s+2)(s+5)}$. Also the block labeled H_5 corresponds with the time-varying gain .25 (1-cost). Finally, the blocks labeled H_4 and H_6 correspond with, respectively, the piecewise linear nonlinearity and the hysterisis nonlinearity. Hence, with $k = .5$ it is seen the system of Fig. 14 fits the form of the system shown in Fig. 15. Now let $\overline{H}_1(s) = \frac{s+20}{(s+1)(s+2)}$, $\overline{H}_2(s) = \frac{4}{(s+2)(s+4)}$, $\overline{H}_3(s) = \frac{s+4}{(s+1)(s+2)(s+5)}$, the relation H_5 satisfy the equation $H_5x(t) = .25(1-\cos t)x(t)$, and the relations H_4 and H_6 be described by the graphs shown in Fig. 14. From the Nyquist diagrams shown in Fig. 16, it is clear from Lemma 1 of Chapter 3 that H_1 is outside $\{-5.33,-.5\}$ and both H_2 and H_3 are inside $\{-.5,.5\}$. Further, it is clear from Lemma 2 of Chapter 3 that H_5 is inside $\{0,.5\}$, H_4 is inside $\{.53,1.6\}$, and H_6 is inside $\{0,.5\}$. Now pick $\delta = .375$. Then for $a_1 = -.5$, $b_1 = -5.33$, $b_2 = b_3 = .5$, and $b_5 = b_6 = .5$

$$\delta(1 - b_2 b_3 b_5) + k b_2 (b_3 + b_6)\frac{2b_1 a_1}{b_1 - a_1} > 0.$$

Further,

$$-\frac{1}{b_1} - \delta\left(\frac{b_1 - a_1}{2b_1 a_1}\right) = .526$$

and

$$-\frac{1}{a_1} + \delta\left(\frac{b_1 - a_1}{2b_1 a_1}\right) = 1.66.$$

Hence, the boundedness conditions cited above for the system of Fig. 15 are satisfied by the system of Fig. 14.

Now consider the modification discussed above of the system of Fig. 14. The modified system is still of the form of Fig. 15 but with the relations H_4 and H_6 satisfying, respectively, the equations $H_4 x(t) = N_1[x(t)]$ and $H_6 x(t) = N_2[x(t)]$. This system is found to be continuous from Theorem 9 by showing the incremental counterparts of the conditions satisfied by the system of Fig. 14 are satisfied here. From Lemma 1, H_1 is incrementally outside $\{-5.33,-.5\}$ and both H_2 and H_3 are incrementally inside $\{-.5,.5\}$. Further, from Lemma 3 of Chapter 3, H_5 is incrementally inside $\{0,.5\}$, H_4 is incrementally inside $\{.53,1.6\}$ and H_6 is incrementally inside $\{0,.5\}$. Hence, the modified system is continuous.

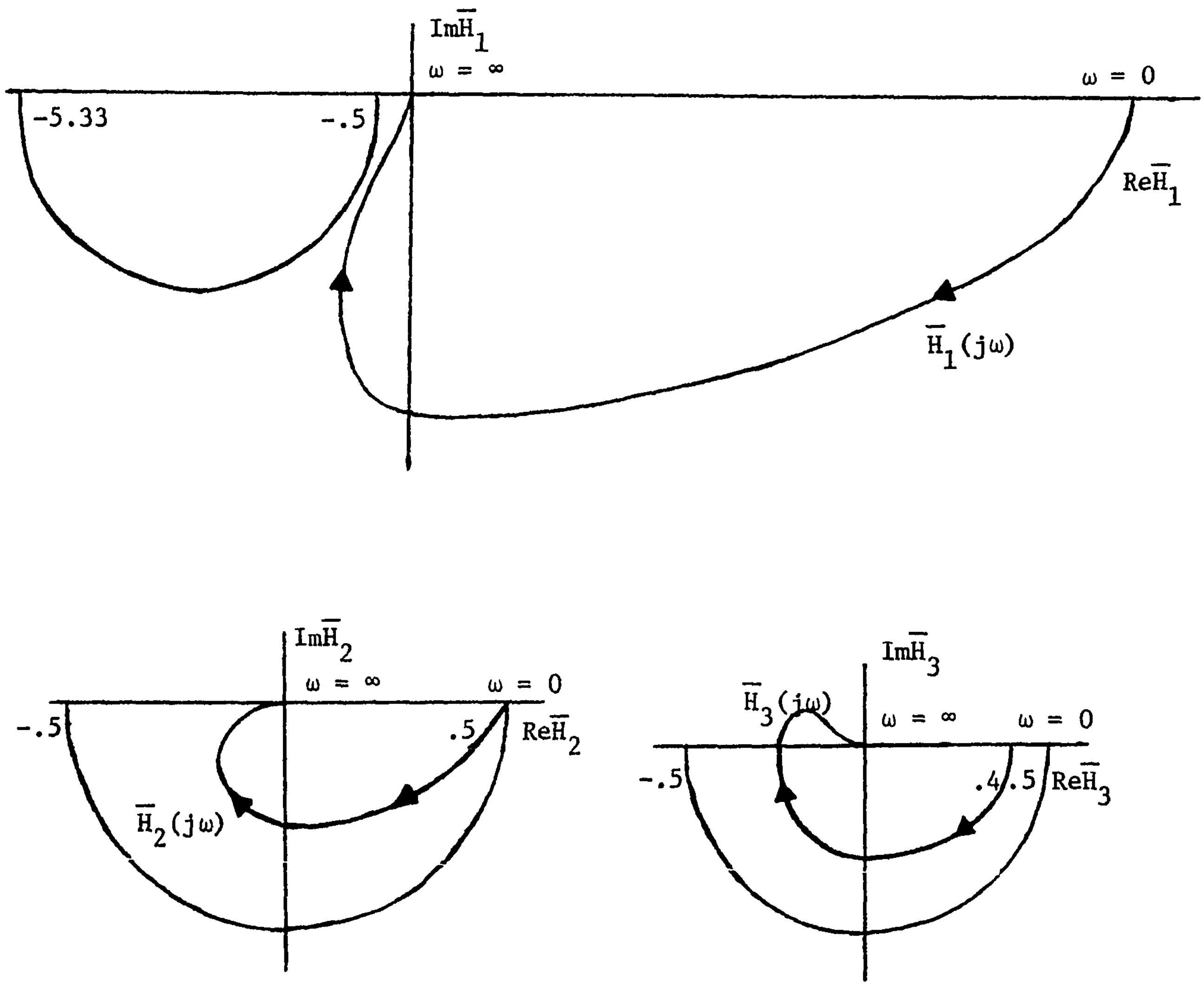


Fig. 16. Nyquist diagrams for Example I.

Now the boundedness conditions stated above for the system of Fig. 15 are verified through the use of Theorem 8. Assume these conditions are satisfied. Then each H_i is conic with constants (a_i,b_i) where

$a_4 = -\frac{1}{b_1} - \delta(\frac{b_1-a_1}{2b_1a_1})$, $b_4 = -\frac{1}{a_1} + \delta(\frac{b_1-a_1}{2b_1a_1})$, $a_2 = -b_2$, $a_3 = -b_3$, and

$a_5 = a_6 = 0$. In Theorem 8 select A = {2,3,4,5,6} and C = {1}. Then

$d_2 = d_3 = 0$, $d_5 = -\frac{b_5}{2}$, and $d_6 = -\frac{b_6}{2}$. Further, $\eta_1 = -(\frac{2b_1a_1}{b_1-a_1})$, $\eta_2 = b_2$, $\eta_3 = b_3$, $\eta_5 = \frac{b_5}{2}$, and $\eta_6 = \frac{b_6}{2}$. Now from the block diagram of Fig. 15 the B matrix is found to be

$$B = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & -1 \\ k & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easily found that

$$(I+B[\text{diag}d_i])^{-1} = \begin{bmatrix} 1 & 0 & 0 & d_4 & 0 & d_6 \\ 0 & 1 & 0 & 0 & d_5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From this, B' is calculated to be

$$B' = (I+B[\text{diag}d_i])^{-1}(B+[\text{diag}c_i]) = \begin{bmatrix} c_1+d_4 & d_6 & -1 & -1 & 0 & -1 \\ k & 0 & d_5 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Due to the fact that the single-loop system possessing open-loop relations H_1 and $-H_4$ has a margin of boundedness δ, it is easily found the element in the upper left hand corner of B' is zero. Also $1 - \eta_1\eta_4 = \delta$.

Now further manipulations give

$$I - \lfloor |b_{ij}'|\eta_j \rfloor = \begin{bmatrix} 1 & \eta_2 d_6 & -\eta_3 & -\eta_4 & 0 & -\eta_6 \\ -k\eta_1 & 1 & \eta_3 d_5 & 0 & -\eta_5 & 0 \\ 0 & -\eta_2 & 1 & 0 & 0 & 0 \\ -\eta_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\eta_3 & 0 & 1 & 0 \\ 0 & -\eta_2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Calculation of the successive principal minors leads to the following five inequalities whose satisfaction guarantees boundedness:

$$1 + k\eta_1\eta_2 d_6 > 0,$$

$$(1 + \eta_2\eta_3 d_5) + k\eta_1\eta_2(d_6 - \eta_3) > 0,$$

$$(1 - \eta_4\eta_1)\ (1 + \eta_2\eta_3 d_5) + k\eta_1\eta_2(d_6 - \eta_3) > 0,$$

$$(1 - \eta_4\eta_1)(1 + 2\eta_2\eta_3 d_5) + k\eta_1\eta_2(d_6 - \eta_3) > 0,$$

$$(1 - \eta_4\eta_1)(1 + 2\eta_2\eta_3 d_5) + k\eta_1\eta_2(2d_6 - \eta_3) > 0.$$

Observing that d_5 and d_6 are negative, it is easily seen the last inequality implies all the ones preceding it. Using the expressions for the η_i and d_i and noting again $1 - \eta_1\eta_4 = \delta$, the last inequality above becomes

$$\delta(1 - b_2b_3b_5) + kb_2(b_3 + b_6)\frac{2b_1a_1}{b_1-a_1} > 0.$$

But this is exactly the condition assumed satisfied at the beginning. Hence, boundedness is obtained.

Example II: Boundedness conditions are given in this example for two types of systems which have the same equations in functional form. First a network formed from the interconnection of a time-varying nonlinear conductance, a time-varying nonlinear resistance, and two passive elements is considered. Then a system is examined which is an interconnection of two linear time-invariant systems and two nonlinearities.

The network to be examined is shown in Fig. 17. The voltages and currents labeled e_i are considered to be inputs to the appropriate elements while the voltages and currents labeled y_i are considered to be outputs. The element labeled H_1 has a current input and a voltage output while the element H_2 has a voltage input and a current output. Because of this, H_1 is referred to as an impedance element and H_2 is referred to as an admittance element. These two elements are assumed to be passive. This means for each element that if i denotes the current through the element and v denotes the voltage drop across the element then $\int_0^t i(\tau)v(\tau)d\tau \geq 0$ for all $t \geq 0$. The elements labeled H_3 and H_4 are, respectively, a time-varying nonlinear conductance and a time-varying nonlinear resistance. These elements are characterized by the functions N_1 and N_2 through the equations

$$y_3(t) = N_1[e_3(t),t]$$

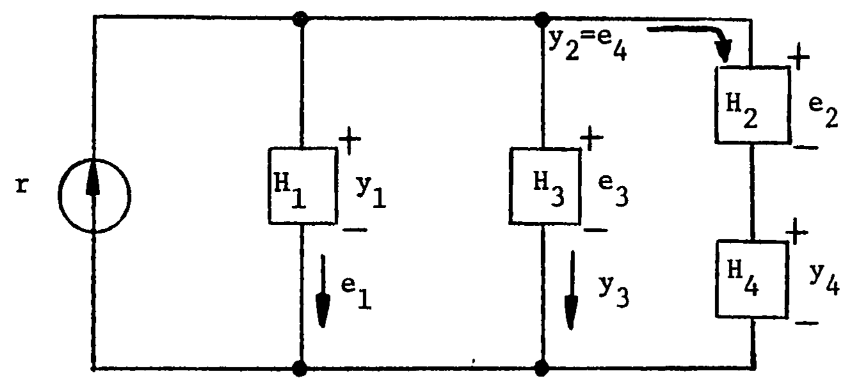


Fig. 17. Network for Example II.

and

$$y_4(t) = N_2[e_4(t),t].$$

It is assumed for each element that all inputs and outputs are square integrable over any finite time-interval. Certainly from an engineering viewpoint this is a trivial assumption.

Now Theorem 8 can be utilized to obtain the following results. Assume for each time t the graph of each of the functions N_1 and N_2 lies within the appropriate shaded region of Fig. 18 for some $\varepsilon > 0$ and $a_3 > 0$ where b_3 and b_4 are arbitrarily large. Being more precise, assume there exist constants $\varepsilon > 0$, $a_3 > 0$, b_3, and b_4 such that the following conditions are satisfied by the functions N_1 and N_2:

$$a_3 \leq \frac{N_1(x,t)}{x} \leq b_3 \text{ for all } x \neq 0 \text{ and all } t\varepsilon[0,\infty),$$

$$N_1(0,t) = 0 \text{ for all } t\varepsilon[0,\infty),$$

$$\frac{1}{a_3} + \varepsilon \leq \frac{N_2(x,t)}{x} \leq b_4 \text{ for all } x \neq 0 \text{ and all } t\varepsilon[0,\infty),$$

$$N_2(0,t) = 0 \text{ for all } t\varepsilon[0,\infty).$$

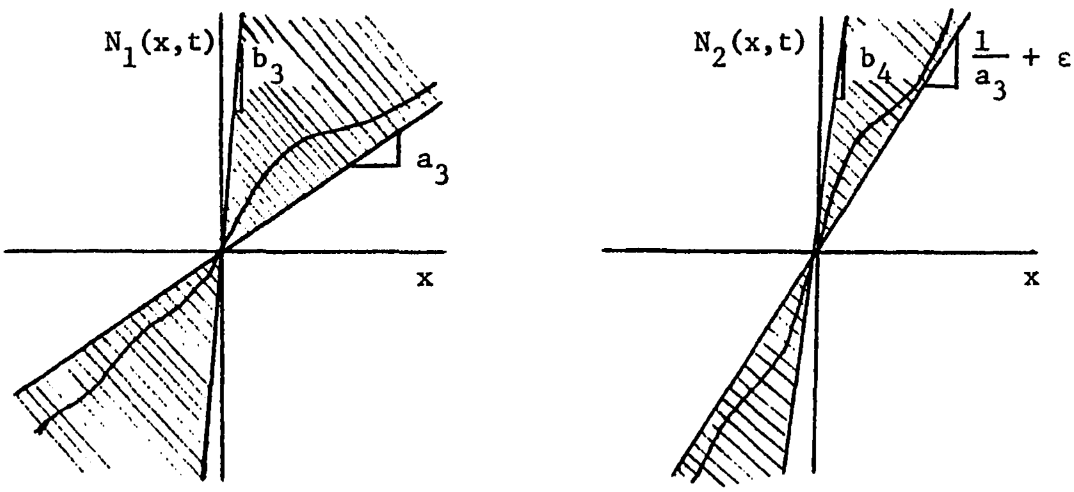


Fig. 18. Conditions on elements H_3 and H_4. (a) Nonlinear conductance. (b) Nonlinear resistance.

Then for each e_i of the network corresponding to a current input r with $\int_0^\infty r^2(t)dt < \infty$ a constant K_i can be calculated so that

$$\int_0^\infty e_i^2(t)dt \leq K_i \int_0^\infty r^2(t)dt.$$

Similarly for each y_i corresponding to an input r with $\int_0^\infty r^2(t)dt < \infty$ a constant L_i can be calculated such that

$$\int_0^\infty y_i^2(t)dt \leq L_i \int_0^\infty r^2(t)dt.$$

It is interesting to observe that by varying the constant a_3 the condition on element H_3 can be relaxed if a more stringent condition is placed on element H_4 and conversely. Hence, a tradeoff in conditions is revealed here.

Now consider the system shown in Fig. 19 formed from the interconnection of two linear systems with two nonlinearities. The nonlinearities are memoryless and characterized by the functions N_1 and N_2.

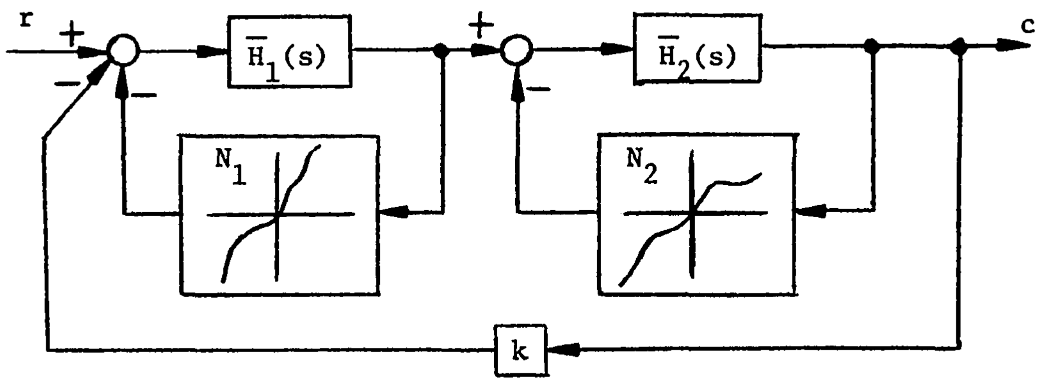


Fig. 19. Block diagram for Example II.

The constant gain k in the outer feedback loop is nonnegative. Systems modeled by differential equations are represented by the blocks labeled $\overline{H}_1(s)$ and $\overline{H}_2(s)$. It is assumed these functions of s are in the form

$$\overline{H}_1(s) = \frac{\sum_{j=0}^{n_1} c_{j1} s^j}{\sum_{j=0}^{n_1} d_{j1} s^j}$$

and

$$\overline{H}_2(s) = \frac{\sum_{j=0}^{n_2} c_{j2} s^j}{\sum_{j=0}^{n_2} d_{j2} s^j}$$

where $d_{n_1 1} \neq 0$ and $d_{n_2 2} \neq 0$. Then it is assumed the systems represented by the blocks labeled $\overline{H}_1(s)$ and $\overline{H}_2(s)$ each have input u and output v which satisfy, respectively, the differential equations

$$\sum_{j=0}^{n_1} d_{j1} v^{(j)} = \sum_{j=0}^{n_1} c_{j1} u^{(j)}$$

and

$$\sum_{j=0}^{n_2} d_{j1} v^{(j)} = \sum_{j=0}^{n_2} c_{j2} u^{(j)} \quad .$$

The superscript denotes differentiation of that order. The two polynomials $\sum_{j=0}^{n_1} d_{j1} s^j$ and $\sum_{j=0}^{n_2} d_{j2} s^j$ each have all zeros strictly in the left half plane. The assumption is made, of course, that each block in Fig. 19 has all inputs and outputs square integrable over any finite time-interval.

Application of Theorem 8 produces the following. Assume there exist constants $b_1 < a_1 < 0$ and $b_2 < a_2 < 0$ such that the loci of $\overline{H}_1(j\omega)$ and $\overline{H}_2(j\omega)$ lie within the appropriate shaded regions of Fig. 20 for $\omega\varepsilon(-\infty,\infty)$ and do not encircle the unshaded regions. Further, assume there exist constants $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$ such that the graphs of the nonlinearities lie within the appropriate shaded region of Fig. 20. Now if

$$\delta_1 \delta_2 > \frac{4b_1a_1b_2a_2k}{(b_1-a_1)(b_2-a_2)} \ ,$$

then in the L_2 sense a bound on the "size" of any input or output of any block of Fig. 19 can be calculated in terms of a bound on the "size" of the input r. Being more specific, consider the output c corresponding

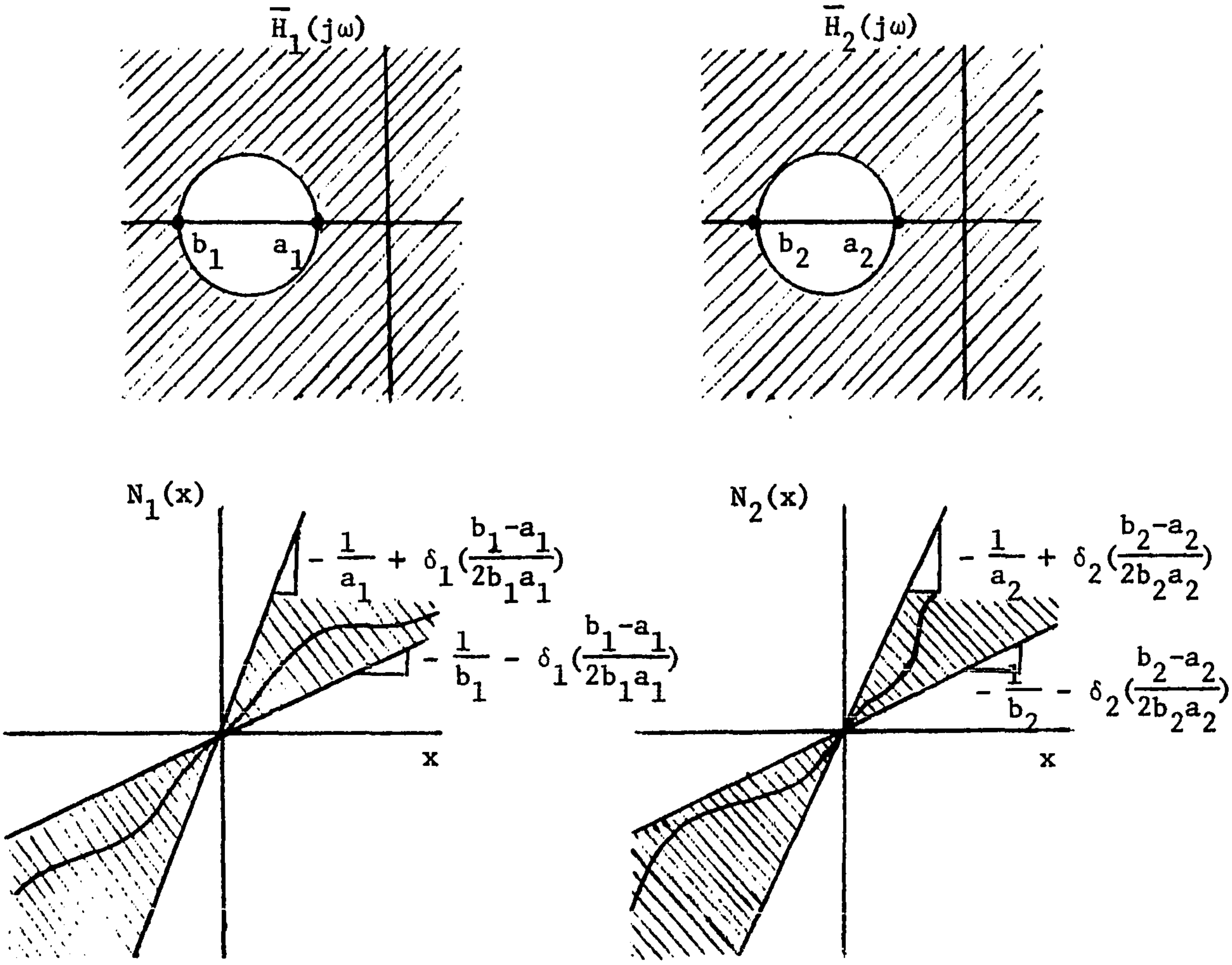


Fig. 20. Graphical conditions imposed on components of system in Fig. 19.

to an input r. For each D and each set of initial conditions there exists a C such that

$$\int_0^\infty r^2(t)dt < D$$

implies

$$\int_0^\infty c(t)^2 dt < C.$$

Loosely speaking, this means an input r which becomes small rapidly enough in the remote future leads to an output c which becomes small in the remote future.

It is interesting to observe the conditions imposed on the system of Fig. 19 are in a form which reveal tradeoffs. For instance, if δ_2 is decreased while increasing δ_1 proportionately, then the conditions remain satisfied. This corresponds to relaxing the condition on the nonlinearity N_2 at the expense of the condition on the nonlinearity N_1.

Now boundedness results in the sense of the L_2 norm are presented for a set of simultaneous functional equations. These results can be shown to specialize to the results given above for the network of Fig. 17 and the system represented in Fig. 19. Consider the equations

$$
\begin{aligned}
e_1 &= r + w_1 - ky_2 - y_3 \\
e_2 &= w_2 + y_1 - y_4 \\
e_3 &= w_3 + y_1 \\
e_4 &= w_4 + y_2 \\
y_i &= H_i e_i \quad \text{for } i = 1, 2, 3, 4
\end{aligned}
\tag{6}
$$

where each H_i is a relation on $L_{2e}[0,\infty)$, each $w_i \varepsilon L_2[0,\infty)$, and the constant k is nonnegative. These equations are clearly of the form of equations (2) where r corresponds to x.

The block diagram corresponding to equations (6) is shown in Fig. 21. This system is clearly in a suitable form for application of Theorem 8. Let subloop 1 denote the loop having relations H_1 and H_3

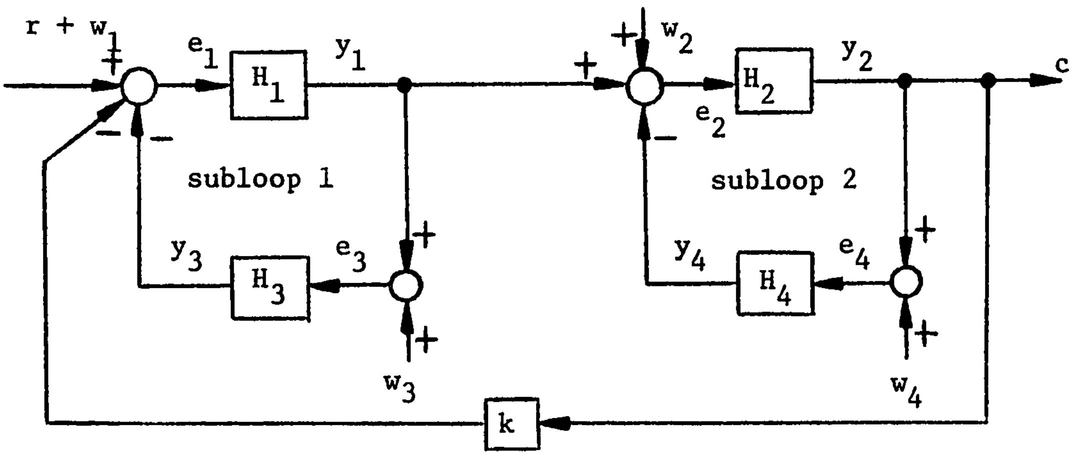


Fig. 21. Block diagram of functional equations for Example II.

and subloop 2 denote the loop having relations H_2 and H_4. From the structure of the interconnection, it seems a reasonable approach is to assume subloop 1 has a margin of boundedness δ_1 and subloop 2 has a margin of boundedness δ_2. Then if the transformed system has the same structure, boundedness conditions will involve the additional loop corresponding to the loop containing H_1, H_2, and k.

Utilizing the above approach leads to the following results. Assume the single-loop system possessing open-loop relations H_1 and $-H_3$ has a margin of boundedness δ_1. Further, assume the single-loop system possessing open-loop relations H_2 and $-H_4$ has a margin of boundedness δ_2. Then boundedness is assured if

$$\delta_1 \delta_2 > \eta_1 \eta_2 k$$

where for i = 1, 2

$$\eta_i = \begin{cases} -\left(\frac{2b_i a_i}{b_i - a_i}\right) & \text{if } H_i \text{ is conic with constants } (a_i, b_i) \\ -2a_i & \text{if } H_i - a_i I \text{ is positive} \end{cases} .$$

In the transformed system the loop containing H_1', H_2', and k corresponds to the loop containing H_1, H_2, and k. This is found true later due to the fact B' = B. The η_1 and η_2 defined above are found from Theorem 8 to be such that H_1' is inside $\{-\eta_1, \eta_1\}$ and H_2' is inside $\{-\eta_2, \eta_2\}$. Hence, the influence of the loop involving H_1', H_2', and k is indicated by the term $\eta_1\eta_2 k$ in the above inequality.

For the special case k = 0 the outer feedback loop is broken so boundedness of the entire system is implied by boundedness of subloops 1 and 2. The above boundedness conditions reflect this by becoming $\delta_1\delta_2 > 0$. But this is already true from the definition of a margin of boundedness δ. Effectively the above conditions reduce to requiring the conditions of Theorem 5 of Chapter 3 be satisfied for both subloop 1 and subloop 2.

Now the results presented for the network of Fig. 17 are found to be a special case of the boundedness results presented for equations (6). First observe that with each w_i set equal to zero and k = 1 the equations of the network are of the same form as equations (6). Assumptions made on the circuit elements guarantee each H_i is a relation on $L_{2e}[0,\infty)$.

Now it is shown conditions satisfied by elements of the network guarantee boundedness. Since H_1 and H_2 are passive, both $\int_0^t e_1(\tau)y_1(\tau)d\tau$

and $\int_0^t e_2(\tau)y_2(\tau)d\tau$ are nonnegative for all $t\varepsilon[0,\infty)$. But in the L_2 space $\int_0^t e_1(\tau)y_1(\tau)d\tau = \langle e_{1t},(H_1e_1)_t\rangle$ and $\int_0^t e_2(\tau)y_2(\tau)d\tau = \langle e_{2t},(H_2e_2)_t\rangle$.

Hence, H_1 and H_2 are both positive relations on $L_{2e}[0,\infty)$. Now, referring to Fig. 18, select $0 < \delta_1 < 1$, $0 < \delta_2 < 1$, $a_1 < 0$, and $a_2 < 0$ such that $-\delta_1(\frac{1}{2a_1}) = a_3$, $-\delta_2(\frac{1}{2a_2}) = \frac{1}{a_3} + \varepsilon$, $-\frac{1}{a_1} + \delta_1(\frac{1}{2a_1}) = b_3$, and $-\frac{1}{a_2} + \delta_2(\frac{1}{2a_2}) = b_4$. This can be done since the first two equalities effectively determine the ratios $\frac{\delta_1}{a_1}$ and $\frac{\delta_2}{a_2}$ while the last two equalities determine particular values of a_1 and a_2. Now since H_1 and H_2 are both positive relations, the relations H_1-a_1I and H_2-a_2I are also both positive. Further, from Lemma 2 of Chapter 3, H_3 is inside $\{-\delta_1(\frac{1}{2a_1}), -\frac{1}{a_1} + \delta_1(\frac{1}{2a_1})\}$ and H_4 is inside $\{-\delta_2(\frac{1}{2a_2}), -\frac{1}{a_2} + \delta_2(\frac{1}{2a_2})\}$. Hence, from Case 2 of the definition of a margin of boundedness δ it is seen subloops 1 and 2 have, respectively, margins of boundedness δ_1 and δ_2. All that is needed now to infer boundedness is that $\delta_1\delta_2 > \eta_1\eta_2 k = 4a_1a_2$. But $-\delta_1(\frac{1}{2a_1}) = a_3$ and $-\delta_2(\frac{1}{2a_2}) = \frac{1}{a_3} + \varepsilon$ implies

$$\delta_1\delta_2 = [1 - \varepsilon\delta_1(\frac{1}{2a_1})]4a_1a_2 > 4a_1a_2.$$

Thus the network is found to be bounded in the sense of the L_2 norm. The specific bounds given on the "size" of each e_i and y_i follow from the manner in which Theorem 8 is proven and the fact that each w_i is zero.

Now it is shown the results given for the system of Fig. 19 are obtained as a special case of the results given for the system of Fig. 21. Let the relations H_3 and H_4 be used to model, respectively, the nonlinearities N_1 and N_2. Further, let the relations H_1 and H_2 model, respectively, the blocks labeled $\bar{H}_1(s)$ and $\bar{H}_2(s)$. From the discussion of the class of operators Q in Chapter 3, it is clear H_1 and H_2 are members of Q having, respectively, Laplace transforms $\bar{H}_1(s)$ and $\bar{H}_2(s)$. Further, from the differential equation models of H_1 and H_2, it is clear initial condition responses are in $L_2[0,\infty)$. The functions w_i can be used to account for initial condition responses and are also in $L_2[0,\infty)$. It is now easily found the system of Fig. 19 has functional equations of the form of equations (6).

Now it is shown conditions guaranteeing boundedness of equations (6) are satisfied under conditions imposed on the system of Fig. 19. From Lemmas 1 and 2 of Chapter 3, it is seen the conditions placed on the Nyquist diagram of H_1 and the graph of N_1 imply H_1 is conic with constants (a_1,b_1) where $b_1 < a_1 < 0$ and H_3 is inside the sector

$$\{-\frac{1}{b_1} - \delta_1(\frac{b_1-a_1}{2b_1a_1}),\ -\frac{1}{a_1} + \delta_1(\frac{b_1-a_1}{2b_1a_1})\}$$

where $0 < \delta_1 < 1$. This implies, from Case 1b of the definition of a margin of boundedness δ, that the single-loop system possessing open-loop relations H_1 and $-H_3$ has a margin of boundedness δ_1. Similarly, it is found the single-loop system possessing open-loop relations H_2 and $-H_4$ has a margin of boundedness δ_2. Now boundedness in the sense of the L_2 norm is assured if

$$\delta_1\delta_2 > \eta_1\eta_2 k = \frac{4b_1a_1b_2a_2k}{(b_1-a_1)(b_2-a_2)} \quad .$$

But this is exactly the condition under which results for the system of Fig. 19 are given.

Now the boundedness results presented for equations (6) are obtained from Theorem 8. Assume the conditions given are satisfied.

Select A = {3,4} and C = {1,2}. The matrix B is found from equations (6) to be

$$B = \begin{bmatrix} 0 & -k & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

Then the matrix B' is easily found to be

$$B' = \begin{bmatrix} c_1+d_3 & -k & -1 & 0 \\ 1 & c_2+d_4 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

Now it is not difficult to find that $c_1 + d_3 = c_2 + d_4 = 0$ due to the fact that subloops 1 and 2 have margins of boundedness δ_1 and δ_2, respectively. Note this means the transformed system has the same form as the original system since B' = B. This leads to

$$I - [|b_{ij}'|\eta_j] = \begin{bmatrix} 1 & -k\eta_2 & -\eta_3 & 0 \\ -\eta_1 & 1 & 0 & -\eta_4 \\ -\eta_1 & 0 & 1 & 0 \\ 0 & -\eta_2 & 0 & 1 \end{bmatrix}.$$

Calculation of the successive principal minors produces the following sufficient conditions for boundedness:

$$1 - k\eta_1\eta_2 > 0,$$

$$(1 - \eta_3\eta_1) - k\eta_1\eta_2 > 0,$$

$$(1 - \eta_3\eta_1)(1 - \eta_4\eta_2) - k\eta_1\eta_2 > 0.$$

Clearly the first two conditions can be eliminated since they are implied by the third. Now, due to the conditions on subloops 1 and 2, it is found that $1 - \eta_3\eta_1 = \delta_1$ and $1 - \eta_4\eta_2 = \delta_2$. Hence, the system is bounded if

$$\delta_1\delta_2 < \eta_1\eta_2 k.$$

But this condition is assumed to be satisfied.

Example III: Some properties of the solutions of the following system of differential equations involving time delay are investigated here:

$$\dot{x}_1(t) = - x_1(t) - .25x_2(t) - 4.5x_3(t) - .25x_1(t-.758)$$

$$- .25N_1[x_1(t-.758),t] + .25r(t)$$

$$\dot{x}_2(t) = -1.25x_2(t) - 4.5x_3(t) - .25x_1(t-.758) - .25N_2[18x_3(t) + x_2(t),t] + .25r(t) \tag{7}$$

$$\dot{x}_3(t) = x_2(t) - 2x_3(t).$$

It is assumed the time-varying nonlinearities N_1 and N_2 are continuous functions of both arguments and that the input r is a continuous function of time.

Strictly speaking, the question of existence of solutions to (7) need not be considered in a stability investigation. However, information concerning this question is usually desired and is readily available in this particular situation. In Halanay [2] it is shown that given a continuous function $v_o(t)$ equal to $x_1(t)$ on $[-.758,0]$ and given values for $x_2(0)$ and $x_3(0)$, a continuous differentiable solution exists for equation (7) on $[0,\infty)$. This solution can be constructed by replacing $x_1(t-.758)$ in (7) by $v_o(t-.758)$ in the time interval $[0,.758]$. By the above continuity assumptions there exists a solution on $[0,.758]$. This solution in turn can be used to produce a solution on $[.758,1.516]$. Repeating this process ad infinitum produces a solution on $[0,\infty)$.

Application of Theorem 8 yields the following results concerning solutions of equations (7). Assume the nonlinearities N_1 and N_2 satisfy the following conditions:

$$2.33x^2 \le xN_1(x,t) \le 5.67x^2 \text{ for all } x \text{ and all } t\varepsilon[0,\infty),$$

$$2.75x^2 \le xN_2(x,t) \le 6.00x^2 \text{ for all } x \text{ and all } t\varepsilon[0,\infty),$$

$$N_1(0,t) = N_2(0,t) = 0 \text{ for all } t\varepsilon[0,\infty).$$

Then for each number D and each set of initial conditions, numbers A_1, A_2, and A_3 can be calculated such that for each input r with

$$\int_0^\infty r^2(t)dt < D$$

a corresponding solution of equations (7) satisfies the inequalities

$$\int_0^\infty x_i^2(t)dt < A_i \text{ for } i = 1,2,3.$$

Theorem 9 can be utilized to obtain further information concerning properties of solutions to equations (7). Let N_1 and N_2 satisfy the following conditions for all x and y and for all $t\varepsilon[0,\infty)$.

$$2.33(x-y)^2 \leq (x-y)[N_1(x,t) - N_1(y,t)] \leq 5.67(x-y)^2,$$

$$2.75(x-y)^2 \leq (x-y)[N_2(x,t) - N_2(y,t)] \leq 6.00(x-y)^2.$$

These conditions are satisfied if, for instance, each N_1 and N_2 is differentiable in the first argument with $2.33 \leq \frac{\partial N_1(x,t)}{\partial x} \leq 5.67$ and $2.75 \leq \frac{\partial N_2(x,t)}{\partial x} \leq 6.00$ for all x and all $t\varepsilon[0,\infty)$. Now let r and r' be inputs satisfying the condition $\int_0^\infty [r(t) - r'(t)]^2 dt < \infty$ and corresponding under identical initial conditions with the, respective, solutions of equations (7) $[x_1(t), x_2(t), x_3(t)]^T$ and $[x_1'(t), x_2'(t), x_3'(t)]^T$. Then constants K_i can be calculated independent of initial conditions such that for i = 1,2,3

$$\int_0^\infty [x_i(t) - x_i'(t)]^2 dt \leq K_i \int_0^\infty [r(t) - r'(t)]^2 dt.$$

Now, in order to obtain the above results, equations (7) must be put in functional form. It is desired to view the nonlinearities N_1 and N_2 as relations on $L_{2e}[0,\infty)$. Hence, expressions are needed for the inputs $x_1(t-.758)$ and $18x_3(t) + x_2(t)$ of these relations. These expressions are obtained from the well-known result of the theory of linear differential equations which gives the solution of the nonhomogeneous problem in terms of solutions of the homogeneous problem.

Let $[x_1(t),\ x_2(t),\ x_3(t)]^T$ be a solution of equations (7) corresponding to the input r for the initial conditions $v_o(t)$ on $[-.758,0]$, $x_2(0)$, and $x_3(0)$. Then from the first equation in (7) it is clear that for $t\varepsilon[0,\infty)$

$$x_1(t) = e^{-t}v_o(0) + \int_0^t e^{-(t-\tau)}[-.25x_2(\tau) - 4.5x_3(\tau) - .25x_1(\tau-.758) - .25N_1\,[x_1(\tau-.758),\tau] + .25r(\tau)]d\tau.$$

Now define the unit step function to be

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} .$$

Referring to the discussion of how a solution of equations (7) is constructed, it is seen that for $t\varepsilon[0,\infty)$

$$x_1(t-.758) = \begin{cases} v_o(t-.758) & \text{for } 0 \leq t \leq .758 \\ e^{-(t-.758)}v_o(0) & \text{for } t \geq .758 \end{cases}$$
$$+ \int_0^t u(t-.758-\tau)e^{-(t-.758-\tau)}[-.25x_2(\tau) - 4.5x_3(\tau) - .25x_1(\tau-.758) - .25N_1[x_1(\tau-.758),\tau] + .25r(\tau)]d\tau. \qquad (8)$$

Now defining the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

it is seen from the last two equations of (7) that for $t\varepsilon[0,\infty)$

$$\begin{bmatrix} x_2(t) \\ x_3(t) \end{bmatrix} = e^{At}\begin{bmatrix} x_2(0) \\ x_3(0) \end{bmatrix} +$$

$$\int_0^t e^{A(t-\tau)}[-.25x_2(\tau) - 4.5x_3(\tau) - .25x_1(\tau-.758)$$

$$- .25N_2[18x_3(\tau) + x_2(\tau),\tau] + .25r(\tau),0]^T d\tau.$$

Calculating e^{At} it is found that

$$18x_3(t) + x_2(t) = (19e^{-t} - 18e^{-2t})x_2(0) + 18e^{-2t}x_3(0)$$

$$+ \int_0^t [19e^{-(t-\tau)} - 18e^{-2(t-\tau)}][-.25x_2(\tau) - 4.5x_3(\tau) \qquad (9)$$

$$-.25x_1(\tau-.758) - .25N_2[18x_3(\tau) + x_2(\tau),\tau] + .25r(\tau)]d\tau.$$

From the above it is clear each solution of equations (7) must satisfy the integral equations (8) and (9). This suggests the following. Let H_1 and H_2 be relations on $L_{2e}[0,\infty)$ which satisfy, respectively, the equations

$$H_1x(t) = N_1[x(t),t]$$

and

$$H_2x(t) = N_2[x(t),t].$$

Since only solutions of equations (7) are of interest, it is assumed that the domains of H_1 and H_2 are each the class of continuous functions. Next, let H_3 and H_4 be operators on $L_{2e}[0,\infty)$ which satisfy, respectively, the equations

$$H_3x(t) = \int_0^t u(t-.758-\tau)e^{-(t-.758-\tau)}x(\tau)d\tau$$

and

$$H_4x(t) = \int_0^t [19e^{-(t-\tau)} - 18e^{-2(t-\tau)}]x(\tau)d\tau.$$

Obviously H_3 and H_4 are both members of the class Q of linear time-invariant operators. Now consider the set of functional equations

$$e_1 = w_1 + y_3$$

$$e_2 = w_2 + y_4$$

$$e_3 = .25r + w_3 - .25y_1 - .25y_3 - .25y_4$$

$$e_4 = .25r + w_4 - .25y_2 - .25y_3 - .25y_4$$

$$y_i = H_ie_i \text{ for } i = 1,2,3,4$$

where each $w_i \varepsilon L_2[0,\infty)$. Clearly these equations are in the form of equations (2).

It is now found for appropriate definitions of each e_i and w_i that solutions of (7) satisfy the above functional equations. Let $[x_1(t),x_2(t),x_3(t)]^T$ be a solution corresponding to the input r and the initial conditions $v_o(t)$ on $[-.758,0]$, $x_2(0)$, and $x_3(0)$. Define $z_1(t)$ and $z_2(t)$ by

$$z_1(t) = \begin{cases} v_o(t-.758) & \text{for } 0 \leq t \leq .758 \\ e^{-(t-.758)} v_o(0) & \text{for } t \geq .758 \end{cases}$$

and

$$z_2(t) = (19e^{-t} - 18e^{-2t})x_2(0) + 18e^{-2t}x_3(0).$$

Then define each e_i and w_i by the following:

$$e_1(t) = x_1(t-.758),$$

$$e_2(t) = 18x_3(t) + x_2(t),$$

$$e_3(t) = -.25x_2(t) - 4.5x_3(t) - .25x_1(t-.758)$$
$$- .25N_1[x_1(t-.758),t] + .25r(t),$$

$$e_4(t) = -.25x_2(t) - 4.5x_3(t) - .25x_1(t-.758)$$
$$-.25N_2[18x_3(t) + x_2(t),t] + .25r(t),$$

$$w_1(t) = z_1(t),$$

$$w_2(t) = z_2(t),$$

$$w_3(t) = w_4(t) = -.25z_1(t) - .25z_2(t).$$

Clearly, since z_1 and z_2 are in $L_2[0,\infty)$, each w_i is in $L_2[0,\infty)$. Now it is seen from equations (8) and (9) that the functional equations are satisfied. For stability purposes only solutions for which r, each e_i, and each y_i belong to $L_{2e}[0,\infty)$ are considered. In this situation it is seen from the earlier discussion of solutions of equations (7) that r, each e_i, and each y_i are continuous. Hence, these functions all belong to $L_{2e}[0,\infty)$. Thus, in a certain sense, the set of solutions of (7) are a subset of the set of solutions of the functional equations. This means stability properties of the functional equations can be used to investigate properties of solutions to equations (7).

Now Theorem 8 is employed to obtain boundedness conditions for the functional equations. The operators H_3 and H_4 have, respectively, the Laplace transforms

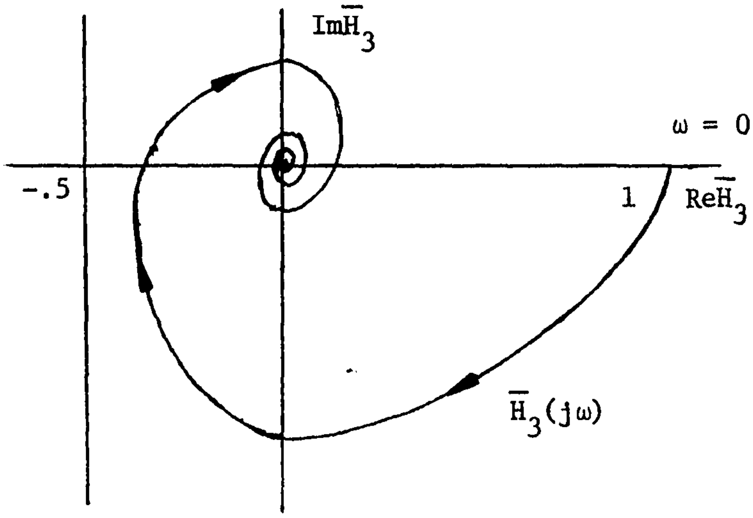
$$\bar{H}_3(s) = \frac{e^{-.758s}}{s+1}$$

and

$$\bar{H}_4(s) = \frac{s+20}{(s+1)(s+2)} .$$

Referring to Lemma 1 of Chapter 3, it is found from the Nyquist diagrams of Fig. 22 that $H_3 + .5I$ is a positive relation and H_4 is outside the sector $\{-5.33,-.5\}$. Hence H_3-a_3I is positive and H_4 is conic with constants (a_4,b_4) where $a_3 = -.5$, $a_4 = -.5$, and $b_4 = -5.33$. Assume for the present that relations H_1 and H_2 are conic with, respectively, constants (a_1,b_1) and (a_2,b_2). Now let $A = \{1,2\}$ and $C = \{3,4\}$. The B matrix is easily found from the functional equations to be

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -.25 & 0 & -.25 & -.25 \\ 0 & -.25 & -.25 & -.25 \end{bmatrix} .$$



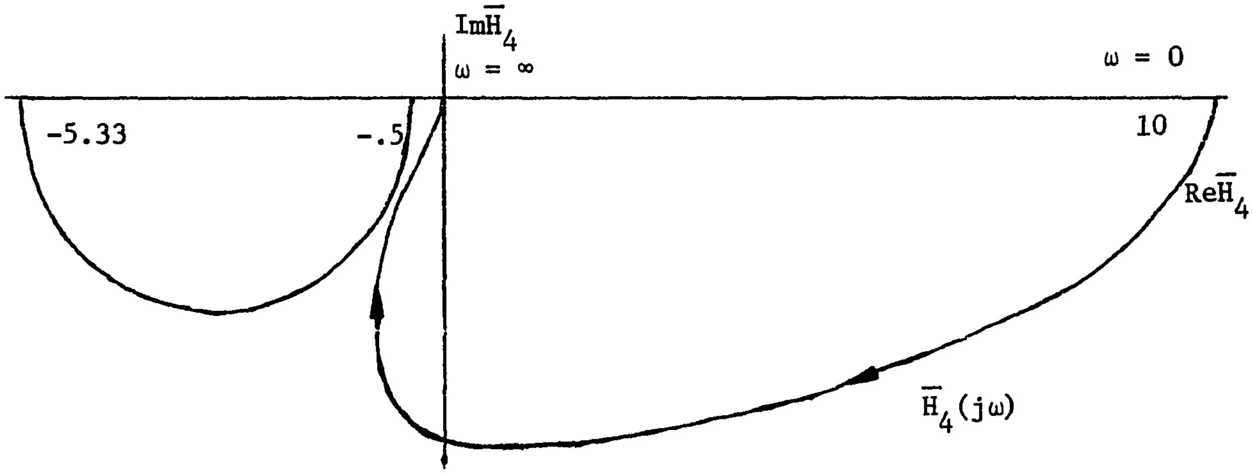


Fig. 22. Nyquist diagrams for Example III.

The B' matrix is given by

$$B' = (I + B[\mathrm{diag}\,d_i])^{-1}(B + [\mathrm{diag}\,c_i]) =$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -.25 & 0 & .25d_1-.25+c_3 & -.25 \\ 0 & -.25 & -.25 & .25d_2-.25+c_4 \end{bmatrix}$$

Now the diagonal terms of this matrix suggest the following. Let the single-loop system possessing open-loop relations H_3 and $-.25H_1$ have a margin of boundedness δ_1. Similarly, let the single-loop system possessing open-loop relations H_4 and $-.25H_2$ have a margin of boundedness δ_2. Note, since the conic nature of H_3 and H_4 is known, specification of δ_1 and δ_2 determines the conic nature of H_1 and H_2. It is easily found that the above conditions imply $.25d_1 + c_3 = .25d_2 + c_4 = 0$. This means the transformed system has the same interconnection structure as the original system since it is now true that $B' = B$. It should be observed at this point that the above assumptions imply $1-.25\eta_1\eta_3 = \delta_1$ and $1-.25\eta_2\eta_4 = \delta_2$.

Now further manipulations produce

$$I - [|b_{ij}'|\eta_j] = \begin{bmatrix} 1 & 0 & -\eta_3 & 0 \\ 0 & 1 & 0 & -\eta_4 \\ -.25\eta_1 & 0 & 1-.25\eta_3 & -.25\eta_4 \\ 0 & -.25\eta_2 & -.25\eta_3 & 1-.25\eta_4 \end{bmatrix} .$$

Calculation of the successive principal minors gives the following boundedness conditions:

$$1 - .25\eta_1\eta_3 - .25\eta_3 > 0,$$

$$(1-.25\eta_1\eta_3)\,(1-.25\eta_2\eta_4) - .25\eta_3(1-.25\eta_2\eta_4) - .25\eta_4(1-.25\eta_1\eta_3) > 0.$$

The first condition can be eliminated since it is implied by the second. Further, from expressions given for η_3 and η_4 in Theorem 8 and Remark 2 and from the above expressions for δ_1 and δ_2, it is found boundedness is guaranteed if

$$\delta_1\delta_2 - .25\delta_2 - .275\delta_1 > 0.$$

From the above, the results cited earlier for the differential equations (7) can be shown true. To see this consider the particular situation of $\delta_1 = .582$ and $\delta_2 = .551$. It is easily found that

$$\delta_1\delta_2 - .25\delta_2 - .275\delta_1 > 0.$$

Hence, the functional equations are bounded. It is seen from the definition of a margin of boundedness δ that it is required $.25H_1$ be inside the sector $\{\delta_1, 2-\delta_1\}$ and $.25H_2$ be inside the sector $\{.187 + .907\delta_2, 2-.907\delta_2\}$. For the above values of δ_1 and δ_2 this means it is required that H_1 be inside $\{2.33, 5.67\}$ and H_2 be inside $\{2.75, 6.00\}$.

Now assume that in equations (7) the nonlinearities N_1 and N_2 satisfy the following conditions:

$$2.33x^2 \leq xN_1(x,t) \leq 5.67x^2 \text{ for all } x \text{ and all } t\varepsilon[0,\infty),$$

$$2.75x^2 \leq xN_2(x,t) \leq 6.00x^2 \text{ for all } x \text{ and all } t\varepsilon[0,\infty),$$

$$N_1(0,t) = N_2(0,t) = 0 \text{ for all } t\varepsilon[0,\infty).$$

Hence, from Lemma 2 of Chapter 3, the relations H_1 and H_2 are inside the, respective, sectors $\{2.33, 5.67\}$ and $\{2.75, 6.00\}$. Now select a set of initial conditions. This corresponds to fixing the w_i functions. Then pick a number D and a continuous input r such that

$$\int_0^\infty r^2(t)dt < D.$$

It follows there exist constants K and L such that for a corresponding solution $[x_1(t),x_2(t),x_3(t)]^T$ of (7)

$$\int_0^\infty e_1{}^2(t)dt = \int_0^\infty x_1(t-.758^2)dt < K$$

and

$$\int_0^\infty e_4{}^2(t)dt = \int_0^\infty [-.25x_2(t) - 4.5x_3(t) - .25x_1(t-.758)$$

$$- .25N_2[18x_3(t) + x_2(t),t] + .25r(t)]^2dt < L.$$

The first inequality implies there exists an A_1 such that

$$\int_o^\infty x_1(t)^2dt < A_1.$$

From the equation given earlier for $[x_2(t),x_3(t)]^T$, it is easily found that

$$x_2(t) = e^{-t}x_2(0) + \int_o^t e^{-(t-\tau)}e_4(\tau)d\tau$$

and

$$x_3(t) = (e^{-t}-e^{-2t})x_2(0) + e^{-2t}x_3(0)$$

$$+ \int_o^t [e^{-(t-\tau)}-e^{-2(t-\tau)}]e_4(\tau)d\tau.$$

Hence, both x_2 and x_3 are in the form of a sum of a fixed function in $L_2[0,\infty)$ with the output of an operator which is in the class Q. It is easily seen that each operator has a Nyquist diagram which lies within a circle in the complex plane with center at the origin. Thus, by Lemma 1 of Chapter 3, it is seen each operator has a finite gain. Since the $||e_4||^2 < L$, it is then clear there exists constants A_2 and A_3 such that

$$\int_0^\infty x_2^2(t)dt < A_2$$

and

$$\int_0^\infty x_3^2(t)dt < A_3.$$

Now assume the nonlinearities N_1 and N_2 satisfy the following conditions for all x and y and for all $t\varepsilon[0,\infty)$:

$$2.33(x-y)^2 \le (x-y)[N_1(x,t) - N_1(y,t)] \le 5.67(x-y)^2,$$

$$2.75(x-y)^2 \le (x-y)[N_2(x,t) - N_2(y,t)] \le 6.00(x-y)^2.$$

Then from Lemmas 1 and 3 it is clear the incremental counterparts of the above boundedness conditions are satisfied. This implies continuity by Theorem 9. Now fix the w_i functions by selecting a set of initial conditions. Next let r and r' be two continuous inputs with corresponding solutions $[x_1(t), x_2(t), x_3(t)]^T$ and $[x_1'(t), x_2'(t), x_3'(t)]^T$, respectively. Assuming

$$\int_0^\infty [r(t) - r'(t)]^2 dt < \infty,$$

it is found using the obvious definitions for e_1' and e_4' that there exists B and C such that

$$\int_0^\infty [e_1(t) - e_1'(t)]^2 dt \le B \int_0^\infty [r(t) - r'(t)]^2 dt$$

and

$$\int_0^\infty [e_4(t) - e_4'(t)]^2 dt \le C \int_0^\infty [r(t) - r'(t)]^2 dt.$$

The first inequality obviously implies there exists a K_1 such that

$$\int_0^\infty [x_1(t) - x_1'(t)]^2 dt \leq K_1 \int_0^\infty [r(t) - r'(t)]^2 dt.$$

Now examine the equations given above for $x_2(t)$ and $x_3(t)$. Since initial conditions are fixed, it is seen that

$$x_2(t) - x_2'(t) = \int_0^t e^{-(t-\tau)}[e_4(\tau) - e_4'(\tau)]d\tau$$

and

$$x_3(t) - x_3'(t) = \int_0^t [e^{-(t-\tau)} - e^{-2(t-\tau)}][e_4(\tau) - e_4'(\tau)]d\tau.$$

Since the above integral operators have finite gain and since $||e_4 - e_4'||^2 \leq C||r - r'||^2$, it is clear there exist constants K_2 and K_3 such that

$$\int_0^\infty [x_2(t) - x_2'(t)]^2 dt \leq K_2 \int_0^\infty [r(t) - r'(t)]^2 dt$$

and

$$\int_0^\infty [x_3(t) - x_3'(t)]^2 dt \leq K_3 \int_0^\infty [r(t) - r'(t)]^2 dt.$$

CHAPTER 6: CONCLUSION

Conditions sufficient to guarantee boundedness or continuity of a multiple-loop nonlinear time-varying system are presented here. Boundedness results are derived in terms of the interconnection structure of the multiple-loop system and in terms of gains of the relations interconnected to form the system. The range of application of these results is greatly expanded through a certain transformation which leads to a result involving sector conditions. Continuity results are found to be available from application of boundedness conditions to a system which relates changes in inputs to changes in outputs. This leads to results identical with boundedness results but with sector conditions replaced by their incremental counterparts.

An interesting illustration of the theory is provided by examining a system formed by the interconnection of an arbitrary number of memoryless nonlinearities with a number of linear time-invariant relations. It is found inputs belonging to a bounded subset of the L_2 space always correspond to outputs which belong to a bounded subset of the L_2 space under the following conditions:

(1) The Nyquist diagram of each linear relation either lies outside an appropriate circle in the complex plane and does not encircle this circle or the Nyquist diagram lies wholly within an appropriate circle.

(2) The graph of each nonlinearity lies in a region of the plane enclosed by two appropriate straight lines passing through the origin.

The exact meaning of appropriate in (1) and (2) is determined by the interconnection structure. By changing only (2), the above boundedness conditions become continuity conditions. Inputs which are arbitrarily "close" in the sense of the L_2 norm lead to outputs arbitrarily "close" in the sense of the L_2 norm if (2) reads: The slope of each nonlinearity has appropriate upper and lower bounds.

In a certain sense the general theory is found capable of providing some feeling of "how stable" a multiple-loop system is. If boundedness conditions are satisfied, then specific bounds on system outputs in terms of bounds on system inputs can be obtained. Further, if continuity conditions are satisfied, then specific bounds on deviations in system outputs in terms of deviations in system inputs are available. If conditions on the multiple-loop system are tightened, then bounds on system responses and deviations in system responses become tighter. Hence, the margin by which stability conditions are satisfied is somewhat of an indication of the "degree of stability" for the system.

Much can be inferred about stability conditions which the theory can provide solely by examining the form of the interconnection structure. Experience indicates the relative positions of the subloops can be used to guide application of the theory. In applications presented here it is found that if the subloops of the transformed system each satisfy stability conditions by a certain margin then conditions for the entire system can be phrased in terms of these margins. This often reveals tradeoffs in conditions on relations which allow stability to be retained.

Several applications of the theory to specific multiple-loop nonlinear time-varying systems are presented. Three particular systems considered are the following:

(1) a certain interconnection of three specific linear time-invariant relations with a linear time-varying relation, a piecewise linear relation, and a hysteresis nonlinearity,

(2) a network formed from a passive impedance, a passive admittance, a nonlinear time-varying resistance, and a nonlinear time-varying conductance,

(3) a third order nonlinear time-varying differential equation involving time delay.

LITERATURE CITED

[1] F. R. Gantmacher, Matrix Theory. New York: Chelsea, 1959.

[2] A. Halanay, Differential Equations. New York: Academic, 1966.

[3] J. M. Holtzman, Nonlinear Systems Theory. Englewood Cliffs, New Jersey: Prentice-Hall, 1970.

[4] L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces. New York: Macmillan, 1964.

[5] J. L. Kelley, General Topology. New York: American, 1955.

[6] A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis. Rochester, New York: Graylock, 1957.

[7] T. L. Saaty, Modern Nonlinear Equations. New York: McGraw-Hill, 1967.

[8] I. W. Sandberg, "A Frequency-Domain Condition for the Stability of Feedback Systems Containing a Single Time-Varying Nonlinear Element," Bell Sys. Tech. J., vol. 43, no. 4, part 2, pp. 1601-1608, July 1964.

[9] __________, "On the Boundedness of Solutions of Nonlinear Integral Equations," Bell Sys. Tech. J., vol. 44, no. 3, pp. 439-453, March 1965.

[10] __________, "On the L_2-Boundedness of Solutions of Nonlinear Functional Equations," Bell Sys. Tech. J., vol. 43, no. 4, part 2, pp. 1581-1599, July 1964.

[11] __________, "Some Results on the Theory of Physical Systems Governed by Nonlinear Functional Equations," Bell Sys. Tech. J., vol. 44, no. 5, pp. 871-898, May-June 1965.

[12] A. E. Taylor, General Theory of Functions and Integration. Waltham, Mass.: Blaisdell, 1965.

[13] J. C. Willems, "The Generation of Lyapunov Functions for Input-Output Stable Systems," SIAM J. Control, vol. 9, no. 1, pp. 105-134, February 1971.

[14] G. Zames, "Nonlinear Time-Varying Feedback Systems--Conditions for L_∞-Boundedness Derived Using Conic Operators on Exponentially Weighted Spaces," Proc. 1965 Allerton Conf., pp. 460-471.

[15] __________, "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems--Part I: Conditions Derived Using Concepts of Loop Gain, Conicity and Positivity," IEEE Trans. on Automatic Control, vol. AC-11, no. 2, pp. 228-238, April 1966.

[16] __________, "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems--Part II: Conditions Involving Circles in the Frequency Plane and Sector Nonlinearities," IEEE Trans. on Automatic Control, vol. AC-11, no. 3, pp. 465-476, July 1966.

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APPENDIX A

Several Linear Spaces

Several types of linear spaces are discussed here. A thorough discussion of these spaces is found in many texts [4], [6], [12]. First a definition is given for a linear space. In this definition R denotes either the field of real numbers or the field of complex numbers. A number belonging to R is referred to as a scalar.

Definition: Let X be a set of elements for which the algebraic operations of addition and scalar multiplication are defined. If $x,y \in X$ then the addition operation produces a unique element of X denoted by $x + y$. Further, if $a \in R$ and $x \in X$ then the scalar multiplication operation produces a unique element of X denoted by ax. The set X along with the two algebraic operations is a linear space if the following are true:

(1) $x + y = y + x$ for all $x,y \in X$.

(2) $x + (y + z) = (x + y) + z$ for all $x,y,z \in X$.

(3) There is a unique element of X denoted by 0 such that $x + 0 = x$ for all $x \in X$.

(4) For each $x \in X$ there exists a unique element of X denoted by $-x$ such that $x + (-x) = 0$.

(5) $a(x + y) = ax + ay$ for all $a \in R$ and all $x,y \in X$.

(6) $(a + b)x = ax + bx$ for all $a,b \in R$ and all $x \in X$.

(7) $a(bx) = (ab)x$ for all $a,b \in R$ and all $x \in X$.

(8) $1x = x$ for all $x \in X$.

(9) $0x = 0$ for all $x \in X$.

In the above definition if R denotes the field of real numbers, then the linear space is referred to as a real linear space. Similarly if R denotes the field of complex numbers, then the linear space is referred to as a complex linear space. It should be noted no distinction is made between the number zero and the zero element of X. Which is being referred to is always clear from context.

Now a special kind of linear space, a normed linear space, is defined.

Definition: A normed linear space is a linear space X on which a real-valued function referred to as the norm is defined. The value of the norm at $x \varepsilon X$ is denoted by $||x||$, and the following properties must be satisfied:

(1) $||x+y|| \leq ||x|| + ||y||$ for all $x,y \varepsilon X$.

(2) $||ax|| = |a| \; ||x||$ for all $a \varepsilon R$ and all $x \varepsilon X$.

(3) $||x|| \neq 0$ if $x \neq 0$.

Now a definition is given for an inner product space.

Definition: A complex linear space X is an inner product space if there exists on XxX a complex-valued function called the inner product. The value of the inner product at $(x,y) \varepsilon XxX$ is denoted by $\langle x,y \rangle$, and the following properties must be satisfied:

(1) $\langle x + y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$ for all $x,y,z \varepsilon X$.

(2) $\langle x,y \rangle = \overline{\langle y,x \rangle}$ for all $x,y \varepsilon X$.

(3) $\langle ax,y \rangle = a\langle x,y \rangle$ for all $a \varepsilon R$ and all $x,y \varepsilon X$.

(4) $\langle x,x \rangle \geq 0$ for all $x \varepsilon X$ and $\langle x,x \rangle \neq 0$ if $x \neq 0$.

For the case of a real linear space, the above definition is different. The only changes required are that the inner product be real-valued and the bar over <y,x> be removed in property (2).

Now L_p spaces which are of interest for stability purposes are defined. A condition which is violated at most on a set of measure zero is referred to as being true almost everywhere.

Definition: For $1 \leq p < \infty$ the L_p space is defined as the space of all real measurable functions $x(t)$ such that $|x(t)|^p$ is Lebesgue integrable over T. The L_∞ space is the space of real measurable functions on T such that for each $x(t)$ there exists an M so that $|x(t)| \leq M$ almost everywhere on T.|

In the above definition no distinction is made between functions which agree almost everywhere. Also the notation $L_p[t_o,\infty)$ or $L_p(-\infty,\infty)$ is often used to denote an L_p space for $T = [t_o,\infty)$ or $T = (-\infty,\infty)$.

Each of the L_p spaces is a normed linear space. If $x \varepsilon L_p$ for finite p, then the norm of x is given by $||x|| = (\int_T |x(t)|^p dt)^{1/p}$. For $p = \infty$ the norm of $x \varepsilon L_\infty$ is the infimum of the set of all M such that $|x(t)| \leq M$ almost everywhere on T. This infimum is called the essential supremum and denoted by $\operatorname{ess\,sup}_{t \varepsilon T} |x(t)|$. Hence for $x \varepsilon L_\infty$ it is seen $||x|| = \operatorname{ess\,sup}_{t \varepsilon T} |x(t)|$.

The L_2 space is distinguished from the other L_p spaces by the fact that it is an inner product space. If $x,y \varepsilon L_2$ the inner product is defined by $\langle x,y \rangle = \int_T x(t)y(t)dt$.

APPENDIX B

Relations

Following Kelley [5] definitions are given here for certain manipulations of relations.

Definition: If H and K are relations on X_e and c is a real constant then:

$$H + K = \{(x,y): x \varepsilon Do(H) \cap Do(K) \text{ and there exist images } Hx \text{ and } Kx \text{ such that } y = Hx + Kx\}.$$

$$cH = \{(x,y): x \varepsilon Do(H) \text{ and there exists an image } Hx \text{ such that } y = c(Hx)\}.$$

$$KH = \{(x,y): \text{there exists } z \text{ such that } (x,z) \varepsilon H \text{ and } (z,y) \varepsilon K\}.$$

$$H^{-1} = \{(x,y): (y,x) \varepsilon H\}.$$

$$I = \{(x,y): x \varepsilon X_e \text{ and } y = x\}.$$

It is of interest to note that despite the fact addition and scalar multiplication are defined, the space of all relations on X_e is not a linear space.

APPENDIX C

Completion of Proof for Theorem 7

It is shown here the hypotheses of Theorem 7 are sufficient to guarantee the matrix $I - [|b_{ij}|g(H_j)]$ has an inverse with all non-negative elements. The following two theorems presented in Gantmacher [1, pp. 66 and 71 of Vol. II] are found to be useful.

Theorem 10: A matrix A having all elements nonnegative always has a nonnegative eigenvalue r such that the moduli of all the eigenvalues of A do not exceed r. To this "maximal" eigenvalue r there corresponds an eigenvector y such that $y \geq 0$ and $y \neq 0$. Further, the adjoint matrix $B(\lambda) = (\lambda I - A)^{-1}|\lambda I - A|$ has all elements nonnegative for $\lambda \geq r$. |

Theorem 11: If a matrix G has all off diagonal elements negative or zero and the successive principal minors are positive, then all principal minors are positive. |

Now, as in Theorem 7, assume the successive principal minors of $I - [|b_{ij}|g(H_j)]$ are all positive. Since the last successive principal minor is the determinant of this matrix, the matrix is nonsingular and has an inverse. Now from Theorem 10 it is clear the matrix $[|b_{ij}|g(H_j)]$ has a "maximal" eigenvalue r. Further, the matrix $(I - [|b_{ij}|g(H_j)])^{-1}$ $|I - [|b_{ij}|g(H_j)]|$ has all elements nonnegative if $r \leq 1$. But $|I - [|b_{ij}|g(H_j)]|$ is the last successive principal minor of

$I - [|b_{ij}|g(H_j)]$ and is positive. Hence, if $r \leq 1$ then $I - [|b_{ij}|g(H_j)]$ has an inverse with all nonnegative elements.

Now it only need be shown that $r \leq 1$. Since r is an eigenvalue, it is found

$$0 = |rI - [|b_{ij}|g(H_j)]| = |\{I - [|b_{ij}|g(H_j)]\} + (r - 1)I|$$

This means 1-r is an eigenvalue of the matrix $I - [|b_{ij}|g(H_j)]$. Now from the theory of matrices it is known the characteristic equation for the n x n matrix B can be written

$$|B - \lambda I| = (-\lambda)^n + \sum_{k=1}^{n} S_k(-\lambda)^{n-k} = 0$$

where each S_k is the sum of all principal minors of order k of the matrix B. Thus, letting $B = I - [|b_{ij}|g(H_j)]$ and $\lambda = 1 - r$ results in

$$(r-1)^n + \sum_{k=1}^{n} S_k(r-1)^{n-k} = 0$$

where each S_k is the sum of all principal minors of order k of $I - [|b_{ij}|g(H_j)]$. But from Theorem 11 it is clear all principal minors of $I - [|b_{ij}|g(H_j)]$ are positive. Hence, each $S_k > 0$. Now it is clear the above characteristic equation cannot be satisfied for $r > 1$. Hence, $r \leq 1$.

APPENDIX D

Completion of Proof for Theorem 8

It is shown here that the conditions imposed on H_i in Theorem 8 are sufficient under the transformation to imply H_i' is inside $\{-\eta_i, \eta_i\}$.

First a few properties of conic relations are listed:

(1) If H is conic with constants (a,b), then for any real number k the relation $H + kI$ is conic with constants $(a + k,\ b + k)$.

(2) If H is conic with constants (a,b) and $k > 0$, then kH is conic with constants (ka,kb). If $k < 0$, then kH is conic with constants (kb,ka).

(3) If H is conic with constants (a,b) where $ab \neq 0$, then H^{-1} is conic with constants $(\frac{1}{b},\frac{1}{a})$. The limiting cases of $b \to \infty$ and $a \to -\infty$ are dealt with rigorously in the following manner. If $H-aI$ is positive where $a \neq 0$, then H^{-1} is conic with constants $(0,\frac{1}{a})$. Further, if $-H + bI$ is positive where $b \neq 0$, then H^{-1} is conic with constants $(\frac{1}{b},0)$.

Similar properties to the above are proven in [15]. However, the notation is somewhat different.

Assume $i \epsilon A$. Then $H_i' = H_i + d_i I$. Since H_i is conic with constants (a_i,b_i), it is seen by property (1) above that H_i' is conic with constants $(a_i + d_i,\ b_i + d_i)$. Now $d_i = -\frac{1}{2}(b_i + a_i)$ results in $a_i + d_i = -\frac{1}{2}(b_i-a_i)$ and $b_i + d_i = \frac{1}{2}(b_i-a_i)$. Since $\eta_i = \frac{1}{2}(b_i-a_i) > 0$, it is found that H_i' is inside $\{-\eta_i,\eta_i\}$.

Now assume $i\epsilon C$. Then $H_i' = (H_i^{-1} + c_iI)^{-1}$. The relation H_i is conic with constants (a_i,b_i) where $a_ib_i \neq 0$ due to the fact that $\eta_i = -(\frac{2b_ia_i}{b_i-a_i}) > 0$. Hence, by property (3) above, H_i^{-1} is conic with constants $\{\frac{1}{b_i}, \frac{1}{a_i}\}$. Now by property (1) the relation $H_i^{-1} + c_iI$ is conic with constants $(\frac{1}{b_i} + c_i, \frac{1}{a_i} + c_i)$. Since $c_i = -(\frac{b_i + a_i}{2b_ia_i})$, it is seen that $\frac{1}{b_i} + c_i = -(\frac{b_i-a_i}{2b_ia_i})$ and $\frac{1}{a_i} + c_i = \frac{b_i-a_i}{2b_ia_i}$. Since $b_i \neq a_i$, property (3) can be used again and reveals that $(H_i^{-1} + c_iI)^{-1}$ is conic with constants $(\frac{2b_ia_i}{b_i-a_i}, -(\frac{2b_ia_i}{b_i-a_i}))$. Now because $\eta_i = -(\frac{2b_ia_i}{b_i-a_i}) > 0$, it is seen that H_i' is inside $\{-\eta_i, \eta_i\}$.

Now consider the remark made following the proof of Theorem 8 concerning the limiting cases for $i\epsilon C$ of $b_i \to \infty$ and $a_i \to -\infty$. The case $b_i \to \infty$ is examined first. Assume $H_i - a_iI$ is positive. Now $H_i' = (H_i^{-1} + c_iI)^{-1}$. The constant $a_i \neq 0$ since $\eta_i = -2a_i > 0$. Hence, by property (3) H_i^{-1} is conic with constants $(0,\frac{1}{a_i})$. By property (1), $H_i^{-1} + c_iI$ is conic with constants $(c_i,\frac{1}{a_i} + c_i)$. Since $c_i = -\frac{1}{2a_i}$, it is clear that $H_i^{-1} + c_iI$ is conic with constants $(-\frac{1}{2a_i}, \frac{1}{2a_i})$. Hence, $(H_i^{-1} + c_iI)^{-1}$ is conic with constants $(2a_i,-2a_i)$. Since $\eta_i = -2a_i > 0$, it is true that H_i' is inside $\{-\eta_i,\eta_i\}$. Similarly it can be shown that if $-H_i + b_iI$ is positive then for the modified definitions of c_i and η_i the relation H_i' is inside $\{-\eta_i, \eta_i\}$.